SUBCONVEXITY FOR MODULAR FORM L-FUNCTIONS IN THE t ASPECT

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Abstract. Modifying a method of Jutila, we prove a t-aspect subconvexity estimate for L-functions associated to primitive holomorphic cusp forms of arbitrary level that is of comparable strength to Good's bound for the full modular group, thus improving on a 36-year-old result. A key innovation in our proof is a general form of Voronoi summation that applies to all fractions, even when the level is not squarefree.

1. Introduction

Let $f \in S_k(\Gamma_0(N), \xi)$ be a primitive holomorphic cusp form of weight k, level N, and nebentypus character ξ. (Here and throughout the paper, "primitive" means that f is a normalized Hecke eigenform in the new space.) Let

$$f(z) = \sum_{n\geq 1} \lambda_f(n)\, n^{(k-1)/2}\, e^{2\pi i n z} \quad \text{for } \Im(z) > 0$$

be the normalized Fourier expansion of f at the cusp ∞. The L-function associated to f is defined by

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_{p \text{ prime}} \Big(1 - \frac{\lambda_f(p)}{p^s} + \frac{\xi(p)}{p^{2s}}\Big)^{-1}$$

for $\Re(s) > 1$, and by analytic continuation on the rest of $\mathbb{C}$.

The analogue of the Lindelöf hypothesis for $L(s,f)$, in the t aspect, is the conjecture that

$$|L(\tfrac{1}{2} + it, f)| \ll_\varepsilon (1 + |t|)^\varepsilon \quad \text{for any } \varepsilon > 0.$$

A standard application of the Phragmén–Lindelöf principle shows that

$$|L(\tfrac{1}{2} + it, f)| \ll_\varepsilon (1 + |t|)^{\frac{1}{2}+\varepsilon}.$$

This is called the convexity estimate for $L(s,f)$ (in the t aspect), and any improvement on the size of the exponent on the right-hand side of the inequality is referred to as a subconvexity estimate.

For $N = 1$, Good [8] showed that

$$|L(\tfrac{1}{2} + it, f)| \ll |t|^{\frac{1}{3}} (\log |t|)^{\frac{5}{6}} \quad \text{for } |t| \geq 2,$$

using the spectral theory of the Laplacian for the modular group to estimate so-called shifted convolution sums. Good's approach implicitly relies on the fact that the Selberg eigenvalue conjecture holds for level 1 [13, Theorem 11.4]. To generalize it to arbitrary level, one would

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have to consider the possibility of exceptional eigenvalues, which could potentially lead to a weaker estimate. There are situations where this numerical weakening can be circumvented; for instance, Lau, Liu, and Ye [17] showed, for a related problem, that the contribution from exceptional eigenvalues can be controlled and causes no harm to their final result. It is possible that a similar analysis could be carried out in the present context.

In this paper, we instead consider a subsequent, more elementary, approach developed by Jutila. Using only Voronoi summation, Farey fractions, and estimates for exponential integrals, Jutila proved (again for $N = 1$) that

$$|L(\tfrac{1}{2}+it, f)| \ll_\varepsilon |t|^{\frac{1}{3}+\varepsilon} \quad \text{for } |t| \geq 2.$$

We generalize Huxley's treatment [12] of Jutila's method and obtain a result for arbitrary level that is essentially as strong as Good's:

Theorem 1.1. *Let $f \in S_k(\Gamma_1(N))$ be a primitive cusp form. Then*

$$|L(\tfrac{1}{2}+it, f)| \ll |t|^{\frac{1}{3}} \log|t| \quad \textit{for } |t| \geq 2,$$

with an implied constant that is polynomial in k and N.

Remark. Combining the resolution of the Sato–Tate conjecture [3] with general estimates for sums of multiplicative functions due to Shiu [21], we can marginally improve the inequality in Theorem 1.1 to

$$|L(\tfrac{1}{2}+it, f)| \ll |t|^{\frac{1}{3}} (\log|t|)^{\frac{8}{9}+\frac{8}{27\pi}}.$$

However, the implied constant need no longer be polynomial in k and N; see Lemma 2.2. One could specify the dependence on k and N more precisely with additional work, but it seems unlikely to be competitive with recent hybrid subconvexity bounds for most ranges of the parameters.

Munshi [19] has recently improved Good's bound for the full modular group, proving that

$$|L(\tfrac{1}{2}+it, f)| \ll_\varepsilon |t|^{\frac{1}{3}-\frac{1}{1200}+\varepsilon}$$

for any $\varepsilon > 0$ when $N = 1$. For $N > 1$, prior to this paper, it was known that

$$|L(\tfrac{1}{2}+it, f)| \ll_\varepsilon |t|^{\frac{1}{2}-\delta+\varepsilon}$$

for any $\varepsilon > 0$ with $\delta = \frac{1-2\theta}{8}$ for any primitive $f \in S_k(\Gamma_1(N))$ by the work of Wu [22] and with $\delta = \frac{1-2\theta}{2(3-2\theta)}$ for $k \geq 4$ by the work of Kuan [16]. Here θ is any exponent toward the Ramanujan-Petersson conjecture. Using $\theta = \frac{7}{64}$, we note that $\frac{1-2\theta}{8} = \frac{25}{256}$ and $\frac{1-2\theta}{2(3-2\theta)} = \frac{25}{178}$.

Our main theorem is an instance of what is commonly referred to as a Weyl-type subconvexity estimate which, in the t aspect, states that $|L(\frac{1}{2}+it)| \ll_\varepsilon |t|^{\frac{m}{6}+\varepsilon}$ for an L-function, $L(s)$, of degree m. Classically such estimates are known for the Riemann zeta-function and Dirichlet L-functions. For degree 2, Good [8] and Meurman [18] proved results of this strength for the L-functions associated to holomorphic modular forms and Maass forms on the full modular group. Theorem 1.1 extends Good's work to arbitrary level (while the analogous extension for Maass forms remains an open problem). For primitive L-functions of higher degree, obtaining a subconvexity estimate in the t aspect of Weyl strength remains elusive. Recently Blomer and Milićević [6] have developed a p-adic analogue of Jutila's argument to prove a subconvexity estimate for $L(s, f \times \chi)$ in the character aspect, for a level

1 form f. In an earlier version of this paper, we predicted that our approach could be combined with theirs to prove an analogous result for general level. Indeed, a result of this type has recently been established by Assing [1].

Our main motivation for establishing Theorem 1.1 is its use in some applications involving estimates for zeros of L-functions. In [4], generalizing a method of Conrey and Ghosh [7], we use Theorem 1.1 to prove quantitative estimates for simple zeros of modular form L-functions of arbitrary conductor. Using similar ideas, we can also prove estimates for the number of distinct zeros of L-functions. This work is currently in preparation.

We conclude the introduction with a brief sketch of the proof. Using an approximate functional equation for $L(\frac{1}{2}+it,f)$ (Lemma 2.1), we reduce the problem to estimating sums of the form $\sum_{n=M_1}^{M_2}\lambda_f(n)\,n^{-it}$, where $M_1\le M_2\le 2M_1$. Next, following Jutila, we break the interval $[M_1,M_2]$ into subintervals on which n^{-it} is well approximated by additive characters $ce^{2\pi i\alpha n}$, where $\alpha\in\mathbb{Q}$ has small denominator. A key novelty in our proof is a generalization of the Voronoi summation formula (Lemma 2.4), which applies to all fractions α. Together with a delicate stationary phase analysis (Proposition 3.1), we thus transform the additive character sums into exponential sums that are more complicated but shorter than those at the start. Finally, we derive a general large sieve inequality (Proposition 3.2) to convert the problem into a certain counting problem for Farey fractions (Lemma 3.1) that was solved by Bombieri and Iwaniec.

The outline of the paper is as follows. After some preliminaries on modular forms in Section 2, we prove Theorem 1.1 in broad strokes following the sketch above in Section 3. We defer the most technical parts of the paper, namely the proofs of Propositions 3.1 and 3.2, until Sections 4 and 5, respectively.

2. Modular forms

In this section, we establish some basic properties of modular forms and their L-functions that are needed in the proof of Theorem 1.1. Throughout this section we take $f\in S_k(\Gamma_0(N),\xi)$ to be a primitive cusp form with Fourier coefficients $\lambda_f(n)$, $\bar{f}\in S_k(\Gamma_0(N),\bar{\xi})$ the dual form with Fourier coefficients $\lambda_{\bar{f}}(n)=\overline{\lambda_f(n)}$, and ϵ_f the root number of $L(s,f)$, satisfying

$$\Lambda(s,f)=\epsilon_f N^{\frac{1}{2}-s}\Lambda(1-s,\bar{f}),$$

where $\Lambda(s,f)=\Gamma_{\mathbb{C}}(s+\frac{k-1}{2})L(s,f)$ and $\Gamma_{\mathbb{C}}(s)=2(2\pi)^{-s}\Gamma(s)$.

The first property that we need is a form of ‘approximate functional equation’ for $L(s,f)$:

Lemma 2.1. *Let $g:(0,\infty)\to\mathbb{R}$ be a smooth function with functional equation $g(x)+g(1/x)=1$ and derivatives decaying faster than any negative power of x as $x\to\infty$. Then*

$$L(\tfrac{1}{2}+it,f)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^{\frac{1}{2}+it}}\,g\Big(\frac{n}{\sqrt{C}}\Big)+\epsilon_f\frac{\Gamma_{\mathbb{C}}(\frac{k}{2}-it)}{\Gamma_{\mathbb{C}}(\frac{k}{2}+it)}\sum_{n=1}^{\infty}\frac{\lambda_{\bar{f}}(n)}{n^{\frac{1}{2}-it}}\,g\Big(\frac{n}{\sqrt{C}}\Big)+O_{\varepsilon,g}\Big(N^{\frac{1}{2}}C^{-\frac{1}{4}+\varepsilon}\Big)$$

for any $\varepsilon>0$, where $C=C(f,t)$ is the analytic conductor, defined by

$$C:=\frac{N}{\pi^2}\Big|\frac{k+1}{2}+it\Big|\Big|\frac{k+3}{2}+it\Big|. \tag{2.1}$$

Proof. This is a special case of a result of Harcos [11, Theorem 2.5]. □

Next, we need upper estimates for $|\lambda_f(n)|$ on average:

Lemma 2.2. *Let $\delta \in \{0, 1-\frac{8}{3\pi}\}$, $\alpha \geq 0$, $x \geq \frac{3}{2}$, and $h \geq 1$. Then*

(i) $\displaystyle\sum_{x<n\leq x+h} |\lambda_f(n)|^2 \ll_{k,N} \max(h, x^{\frac{3}{5}})$;

(ii) $\displaystyle\sum_{x<n\leq x+h} |\lambda_f(n)| \ll_{\delta,k,N} \max(h, x^{\frac{3}{5}})(\log x)^{-\delta}$;

(iii) *for $\alpha > 1$,* $\displaystyle\sum_{n>x} |\lambda_f(n)|\, n^{-\alpha} \ll_{\delta,\alpha,k,N} x^{1-\alpha}(\log x)^{-\delta}$;

(iv) *for $\alpha < 1$,* $\displaystyle\sum_{n\leq x} |\lambda_f(n)|\, n^{-\alpha} \ll_{\delta,\alpha,k,N} x^{1-\alpha}(\log x)^{-\delta}$.

Moreover, when $\delta = 0$, the implied constants are polynomial in k and N.

Proof. In his work introducing the Rankin–Selberg method, Rankin [20] proved the estimate

$$\sum_{n\leq x} |\lambda_f(n)|^2 = A_f x + O_{k,N}(x^{\frac{3}{5}}),$$

for a certain explicit $A_f > 0$. One can see that both A_f and the implied constant above grow at most polynomially in k and N, and (i) follows.

As for (ii), when $\delta = 0$, Cauchy's inequality and (i) imply that

$$\sum_{x<n\leq x+h} |\lambda_f(n)| \ll_{k,N} \sqrt{h \max(h, x^{\frac{3}{5}})} \leq \max(h, x^{\frac{3}{5}}),$$

again with a polynomial implied constant. For $\delta = 1 - \frac{8}{3\pi}$, it follows from the resolution of the Sato–Tate conjecture [3] that[1]

$$\sum_{p\leq x} \frac{|\lambda_f(p)|}{p} \leq (1 - \delta + o_{k,N}(1)) \log\log x \quad \text{as } x \to \infty.$$

Inserting this into Shiu's estimate [21, Theorem 1], for any fixed $\varepsilon > 0$ we derive that

$$\sum_{x<n\leq x+h} |\lambda_f(n)| \ll_{\varepsilon,k,N} h(\log x)^{-\delta} \quad \text{uniformly for } h \geq x^{\varepsilon},$$

which is clearly sufficient for (ii).

Note that (ii) implies $\sum_{n\leq x} |\lambda_f(n)| \ll_{\delta,k,N} x(\log x)^{-\delta}$. Using this, a simple exercise in partial summation implies (iii) and (iv). $\square$

Finally, we require a form of Voronoi/Wilton summation. As this name is usually understood, such a formula exists for every fraction $\alpha \in \mathbb{Q}$ only when the level is squarefree. Since we do not want to impose such a restriction on f in our hypotheses, we prove a generalization, the basic idea of which is to replace additive characters by multiplicative characters at finitely many bad primes. To this end, for any Dirichlet character $\chi \pmod q$, let f^χ denote the unique primitive cusp form whose Fourier coefficients $\lambda_{f^\chi}(n)$ satisfy $\lambda_{f^\chi}(n) = \lambda_f(n)\chi(n)$ for all n coprime to q; this is guaranteed to exist by [2, Theorem 3.2] and has level dividing Nq^2.

[1] Equality holds in this estimate when $k > 1$ and f does not have CM. The remaining cases must be handled separately, but are easier to prove and lead to slightly improved estimates. Specifically, $8/(3\pi)$ can be replaced by $2/\pi$ for CM forms, at most $2/3$ for dihedral forms, $5/6$ for tetrahedral forms, $(5+3\sqrt{2})/12$ for octahedral forms, and $(11+6\sqrt{5})/30$ for icosahedral forms.

Lemma 2.3. *Let $\alpha = a/q \in \mathbb{Q}$, and define $q^* = \prod_{p|q} p^{1+\mathrm{ord}_p q}$. Then*

$$\sum_{n=1}^{\infty} \frac{\lambda_f(n)e(\alpha n)}{n^s} = \sum_{\chi \,(\mathrm{mod}\, q)} \sum_{m \left| \left(\frac{\mathrm{lcm}(Nq,q^2)}{\mathrm{cond}(f^\chi)}, q^* \right)\right.} \frac{C(f,\alpha,m,\chi)}{m^s} L(s,f^\chi),$$

for some numbers $C(f,\alpha,m,\chi) \in \mathbb{C}$ satisfying $C(f,\alpha,m,\chi) \ll_q 1$.

Proof. Let us first assume that $q = p^e$ is a prime power and r is a positive integer coprime to p. Then the additive twist of $r^{-s}L(s,f)$ by $\alpha = a/q$ equals

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\lambda_f(n)}{(rn)^s} e\!\left(\frac{arn}{p^e}\right) &= \sum_{k=0}^{e-1} \frac{\lambda_f(p^k)}{(rp^k)^s} \sum_{(n,p)=1} \frac{\lambda_f(n)}{n^s} e\!\left(\frac{arn}{p^{e-k}}\right) + \sum_{k=e}^{\infty} \frac{\lambda_f(p^k)}{(rp^k)^s} \sum_{(n,p)=1} \frac{\lambda_f(n)}{n^s} \\
&= \sum_{k=0}^{e-1} \frac{\lambda_f(p^k)}{(rp^k)^s} \sum_{\chi \,(\mathrm{mod}\, p^{e-k})} \frac{\tau(\overline{\chi})\chi(ar)}{\varphi(p^{e-k})} \sum_{(n,p)=1} \frac{\lambda_f(n)\chi(n)}{n^s} + \sum_{k=e}^{\infty} \frac{\lambda_f(p^k)}{(rp^k)^s} \sum_{(n,p)=1} \frac{\lambda_f(n)}{n^s} \\
&= \sum_{k=0}^{e-1} \frac{\lambda_f(p^k)}{(rp^k)^s} \sum_{\chi \,(\mathrm{mod}\, p^{e-k})} \frac{\tau(\overline{\chi})\chi(ar)}{\varphi(p^{e-k})} E_{f^\chi,p}(p^{-s}) L(s,f^\chi) \\
&\qquad + \left(r^{-s} - E_{f,p}(p^{-s}) \sum_{k=0}^{e-1} \frac{\lambda_f(p^k)}{(rp^k)^s} \right) L(s,f),
\end{aligned}$$

where $E_{f,p}$ and $E_{f^\chi,p}$ denote the Euler factor polynomials of f and f^χ at p, respectively. Note that this is a linear combination of terms of the form $(rp^j)^{-s}L(s,f^\chi)$ for characters $\chi \,(\mathrm{mod}\, p^e)$.

In the general case, by partial fractions, we may express α as a sum of fractions of the form a/p^e, and applying the prime power case inductively yields a decomposition of the required type. The estimate $C(f,\alpha,m,\chi) \ll_q 1$ follows from the fact that the coefficients in the above polynomials depend only on local data of f^χ, together with Deligne's bound. It remains only to be seen that the values of m that occur must divide $\left(\frac{\mathrm{lcm}(Nq,q^2)}{\mathrm{cond}(f^\chi)}, q^*\right)$, which reduces to the following two assertions in the prime power case:

$$k + \deg E_{f^\chi,p} \le e + 1 \tag{2.2}$$

and

$$k + \deg E_{f^\chi,p} + \mathrm{ord}_p \,\mathrm{cond}(f^\chi) \le e + \max(e, \mathrm{ord}_p N). \tag{2.3}$$

Since $k \le e - 1$ and $\deg E_{f^\chi,p} \le 2$, the assertion in (2.2) is clear. As for (2.3), since χ is a character mod p^{e-k}, it follows from [2, Theorem 3.1] that

$$\mathrm{ord}_p \,\mathrm{cond}(f^\chi) \le e - k + \max(e - k, \mathrm{ord}_p N),$$

so (2.3) holds when $\deg E_{f^\chi,p} = 0$. In particular, this is the case when the local constituent π_p of the automorphic representation associated to f is supercuspidal. If π_p is special then we might have $\deg E_{f^\chi,p} = 1$, but this happens only when $\mathrm{ord}_p \,\mathrm{cond}(f^\chi) = 1$, in which case the left-hand side of (2.3) is at most $k + 2 \le e + 1 \le 2e$. Finally, suppose that π_p is a principal series, say $\pi_p = \pi(\chi_1, \chi_2)$. If $\deg E_{f^\chi,p} = 2$ then $\mathrm{ord}_p \,\mathrm{cond}(f^\chi) = 0$, so again we get the upper bound $k+2 \le 2e$. If $\deg E_{f^\chi,p} = 1$ then we may assume that $\chi\chi_1$ is unramified, so that

$$\operatorname{ord}_p \operatorname{cond}(f^\chi) = \operatorname{ord}_p \operatorname{cond}(\chi\chi_2) \le \operatorname{ord}_p \operatorname{cond}(\chi_1) + \operatorname{ord}_p \operatorname{cond}(\chi_2) = \operatorname{ord}_p N,$$

and the left-hand side of (2.3) is at most $e + \operatorname{ord}_p N$. $\square$

Lemma 2.4. *Let $a/q \in \mathbb{Q}$, and let $F : (0,\infty) \to \mathbb{C}$ be a C^2 function of compact support. Define*

$$N_1 = (N,q), \quad N_2 = \frac{N}{N_1}, \quad q_2 = (N_2^\infty, q), \quad q_1 = \frac{q}{q_2},$$

and write

$$\frac{a}{q} = \frac{a_1}{q_1} + \frac{a_2}{q_2},$$

with the fractions on the right-hand side in lowest terms. Then

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_f(n) e\left(\frac{an}{q}\right) F(n) &= \sum_{\chi \ (\mathrm{mod}\ N_1)} \sum_{r \mid N_2 q_2^2} c(f, a/q, r, \chi) \sum_{n=1}^{\infty} \lambda_{\bar{f}^\chi}(n) e\left(-\frac{\overline{a_1 r} n}{q_1}\right) \\ &\qquad \cdot 2\pi i^k \int_0^\infty F(q_1 x) J_{k-1}\left(4\pi\sqrt{\frac{nx}{q_1 r}}\right) \mathrm{d}x,\end{aligned} \tag{2.4}$$

where $\overline{a_1 r}$ denotes a multiplicative inverse of $a_1 r \pmod{q_1}$, and $c(f, a/q, r, \chi) \ll_{q_2} 1$.

Remark. Note that $q_2 \mid N_1$, so that both r and the coefficients $c(f, a/q, r, \chi)$ are $O_N(1)$, independent of q.

Proof. Since $q_2 \mid N_1$, we have $\operatorname{lcm}(Nq_2, q_2^2) = Nq_2$, so applying Lemma 2.3 with $\alpha = a_2/q_2$, we get

$$\lambda_f(n) e\left(\frac{a_2 n}{q_2}\right) = \sum_{\chi \ (\mathrm{mod}\ q_2)} \sum_{m \left| \left(\frac{Nq_2}{\operatorname{cond}(f^\chi)}, n\right)\right.} C(f, a_2/q_2, m, \chi) \lambda_{f^\chi}\left(\frac{n}{m}\right),$$

for some numbers $C(f, a_2/q_2, m, \chi) \in \mathbb{C}$.

Next we compute the sum

$$\sum_{\substack{n \ge 1 \\ m \mid n}} \lambda_{f^\chi}\left(\frac{n}{m}\right) e\left(\frac{a_1 n}{q_1}\right) F(n) = \sum_{n=1}^{\infty} \lambda_{f^\chi}(n) e\left(\frac{a_1 m n}{q_1}\right) F(mn)$$

by applying the Voronoi summation formula [15, Theorem A.4]. Let us first suppose that F is smooth, which is a hypothesis of loc. cit. Put $g = f^\chi$ and $D = \operatorname{cond}(g)$, so that $g \in S_k(\Gamma_0(D), \psi)$, where $\psi = \xi\chi^2$. Set $D_1 = (D, q_1) = N_1/q_2$, $D_2 = D/D_1$, and split the character ψ as a product $\psi_{D_1}\psi_{D_2}$ of characters modulo D_1 and D_2, respectively. Then [15, Theorem A.4] yields

$$\begin{aligned}&\sum_{n=1}^{\infty} \lambda_g(n) e\left(\frac{a_1 m n}{q_1}\right) F(mn) \\ &= \overline{\psi_{D_1}(a_1 m)} \psi_{D_2}(-q_1) \frac{\eta_g(D_2)}{\sqrt{D_2}} \sum_{n=1}^{\infty} \lambda_{g^{\bar{\psi}_{D_2}}}(n) e\left(-\frac{\overline{a_1 m D_2}}{q_1}\right) \int_0^\infty F(mq_1 x) J_{k-1}\left(4\pi\sqrt{\frac{nx}{q_1 D_2}}\right) \mathrm{d}x,\end{aligned}$$

where $\eta_g(D_2)$ is a constant of modulus 1 and $\overline{a_1 m D_2}$ is an inverse of $a_1 m D_2 \pmod{q_1}$. Note that $g^{\overline{\psi}_{D_2}} = \bar{g}^{\psi_{D_1}}$, where $\bar{g} \in S_k(\Gamma_0(D), \overline{\psi})$ is the dual of g. Since D_1 is coprime to the modulus of χ, we further have $\psi_{D_1} = \xi_{D_1} = \xi_{N_1/q_2}$, so that $\bar{g}^{\psi_{D_1}} = \bar{f}^{\overline{\chi}\xi_{N_1/q_2}}$. Since m is restricted to the divisors of Nq_2/D, we see that mD_2 divides $Nq_2/D_1 = N_2 q_2^2$. Writing $r = mD_2$ and making the change of variables $x \mapsto x/m$, the last line becomes

$$\overline{\psi_{D_1}(a_1 m)}\psi_{D_2}(-q_1)\frac{\eta_g(D_2)}{m\sqrt{D_2}}\sum_{n=1}^{\infty}\lambda_{\bar{f}^{\overline{\chi}\xi_{N_1/q_2}}}(n)e\left(-\frac{\overline{a_1 r}n}{q_1}\right)\int_0^{\infty}F(q_1 x)J_{k-1}\left(4\pi\sqrt{\frac{nx}{q_1 r}}\right)\mathrm{d}x.$$

From the estimate provided by Lemma 2.3, we have

$$\overline{\psi_{D_1}(a_1 m)}\psi_{D_2}(-q_1)\frac{\eta_g(D_2)}{m\sqrt{D_2}}C(f, a_2/q_2, m, \chi) \ll_{q_2} 1.$$

Making the change of variables $\chi \mapsto \xi_{N_1/q_2}\overline{\chi}$ yields (2.4).

It remains only to see that (2.4) is valid if F is merely C^2 and not necessarily smooth. Making the substitution $x = q_1 r(\frac{u}{4\pi})^2$, we have

$$\int_0^{\infty}F(q_1 x)J_{k-1}\left(4\pi\sqrt{\frac{nx}{q_1 r}}\right)\mathrm{d}x = \frac{q_1 r}{8\pi^2}\int_0^{\infty}u^{-k}F\left(\frac{q_1^2 r u^2}{16\pi^2}\right)u^k J_{k-1}(\sqrt{n}u)\,\mathrm{d}u.$$

Applying integration by parts twice and using the estimates

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^k J_k(x)\right\} = x^k J_{k-1}(x), \quad \frac{\mathrm{d}}{\mathrm{d}x}\left\{x^{k+1}J_{k+1}(x)\right\} = x^{k+1}J_k(x), \quad \text{and} \quad J_{k+1}(x) \ll_k \frac{1}{\sqrt{x}},$$

we see that this integral is $O(n^{-\frac{5}{4}})$. Thus, the sum over n on the right-hand side of (2.4) is absolutely convergent, and the lemma follows by a standard argument using smooth approximations of F. □

3. Proof of Theorem 1.1

3.1. **Initial reduction.** Let f be as in the statement of Theorem 1.1, and let $t \in \mathbb{R}$. By replacing f with $\bar{f}$ if necessary, we may assume without loss of generality that $t \geq 0$. We may further assume that

$$t \geq \max\left\{k^{\frac{3}{2}}\log k, N^{\frac{3}{2}}, t_0\right\} \tag{3.1}$$

for a large constant t_0, as otherwise the convexity bound implies Theorem 1.1.

Let C denote the analytic conductor defined in (2.1), and fix, for the remainder of the paper, a choice of $\delta \in \{0, 1 - \frac{8}{3\pi}\}$. With $\delta = 0$, all implied constants depend at most polynomially on k and N. In this section, we prove that

$$|L(\tfrac{1}{2}+it, f)| \ll \sum_M \frac{1}{\sqrt{M}}\max_{M_1\in[M,2M]}\left|\sum_{M_1\leq n\leq 2M}\lambda_f(n)\,n^{-it}\right| + O_{k,N}\left(\sqrt{M_0}\,(\log M_0)^{-\delta}\right) \tag{3.2}$$

for any integer $M_0 \in [2, \sqrt{C}]$, where M runs through numbers of the form $2^K M_0$ for integers $K \in [0, \log_2\frac{\sqrt{C}}{M_0} + 1]$. Therefore, in order to prove Theorem 1.1, it suffices to estimate exponential sums of the form

$$\sum_{M_1\leq n\leq M_2}\lambda_f(n)\,n^{-it} \tag{3.3}$$

where $M_1 \le M_2 \le 2M_1$.

Our starting point for the proof of (3.2) is the approximate functional equation for $L(\frac{1}{2} + it, f)$ in the form of Lemma 2.1. We remark that, without loss of generality, we may suppose that the test function g appearing there is decreasing and supported on the interval $[0, 2]$. For example, the function

$$g(x) = \begin{cases} 1, & \text{if } x < \frac{1}{2}, \\ \alpha \displaystyle\int_{\log_2 x}^{1} e^{-\frac{1}{1-t^2}}\,\mathrm{d}t, & \text{if } \frac{1}{2} \le x \le 2, \\ 0, & \text{if } x > 2, \end{cases}$$

where $\alpha = e^{\frac{1}{2}}/(K_1(\frac{1}{2}) - K_0(\frac{1}{2})) = 2.25228\ldots$ is chosen so that $g(\frac{1}{2}) = 1$, has these properties ($K_n(z)$ denotes the usual K-Bessel function). Since $|\epsilon_f \Gamma_{\mathbb{C}}(\frac{k}{2} - it)/\Gamma_{\mathbb{C}}(\frac{k}{2} + it)| = 1$, by (3.1) we have

$$|L(\tfrac{1}{2} + it, f)| \le 2 \left| \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{\frac{1}{2}+it}}\, g\!\left(\frac{n}{\sqrt{C}}\right) \right| + O(1)\,.$$

By the triangle inequality, since $0 \le g(x) \le 1$, we have

$$\left| \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{\frac{1}{2}+it}}\, g\!\left(\frac{n}{\sqrt{C}}\right) \right| \le \sum_{n \le M_0} \frac{|\lambda_f(n)|}{\sqrt{n}} + \sum_{M} \left| \sum_{M < n \le 2M} \frac{\lambda_f(n)}{n^{\frac{1}{2}+it}}\, g\!\left(\frac{n}{\sqrt{C}}\right) \right|, \tag{3.4}$$

with $M_0 \le \sqrt{C}$ and $M = 2^K M_0$ as above. Note that only finitely many values of M are relevant, since $\operatorname{supp}(g) \subseteq [0, 2]$. We will choose M_0 (depending on t) at the end of the proof, but to fix ideas, we note that

$$\left(\frac{t}{\log t}\right)^{\frac{2}{3}} \ll M_0 \ll t^{\frac{2}{3}}. \tag{3.5}$$

Applying Lemma 2.2(iv), we have

$$\sum_{n \le M_0} \frac{|\lambda_f(n)|}{\sqrt{n}} \ll_{k,N} \sqrt{M_0}(\log M_0)^{-\delta}.$$

We now simplify the second sum on the right-hand side of (3.4) using [12, Lemma 5.1.1]. Defining $G(x) = \frac{1}{\sqrt{x}} g(\frac{x}{\sqrt{C}})$, we may assume that G is decreasing, and hence

$$\left| \sum_{M < n \le 2M} \frac{\lambda_f(n)}{n^{\frac{1}{2}+it}}\, g\!\left(\frac{n}{\sqrt{C}}\right) \right| \ll \frac{1}{\sqrt{M}} \max_{M_1 \in [M, 2M]} \left| \sum_{M_1 \le n \le 2M} \lambda_f(n)\, n^{-it} \right|.$$

Combining estimates yields (3.2).

3.2. **Farey fractions and stationary phase.** We now turn our attention to estimating the sum in (3.3). Let M be a size parameter and suppose that $M_0 \le M \ll M_1 \le M_2 \le 2M_1 \ll M$. Define

$$R = \sqrt{\frac{M}{M_0}} \quad \text{and} \quad H := \left\lceil \frac{M^2}{R^2 t} \right\rceil,$$

and let

$$\mathcal{F}(R) = \left\{ \frac{u}{v} : u, v \in \mathbb{Z}, (u, v) = 1, 0 < v \le R \right\}$$

denote the set of extended Farey fractions with denominator less than or equal to R. Consider the interval

$$\Big[-\frac{t}{2\pi(M_1+2H)},-\frac{t}{2\pi(M_2-2H)}\Big]$$

and denote the elements of $\mathcal{F}(R)$ in this interval by α_j, $j=1,\ldots,J$, in increasing order. (If $M_2-M_1<4H$ or if this interval contains no elements of $\mathcal{F}(R)$ then (3.3) is trivially bounded by the error term in (3.9), below. Hence we may assume that this is not the case.) We make the labeling $\alpha_j=-\frac{u_j}{v_j}$, where $u_j,v_j\in\mathbb{Z}_{>0}$, $(u_j,v_j)=1$, and $v_j\le R$.

For consecutive Farey fractions $\alpha_j=-\frac{u_j}{v_j}$ and $\alpha_{j+1}=-\frac{u_{j+1}}{v_{j+1}}$ the mediant, denoted ρ_j, is given by $\rho_j=-\frac{u_j+u_{j+1}}{v_j+v_{j+1}}$. Note that

$$|\rho_j-\alpha_j|=\frac{1}{v_j(v_j+v_{j+1})}\asymp\frac{1}{v_jR} \tag{3.6}$$

and similarly $|\rho_j-\alpha_{j+1}|\asymp\frac{1}{v_jR}$. Define the function $h(y)=-\frac{t}{2\pi y}$ and integers $\mathcal{N}_0=M_1+2H$, $\mathcal{N}_J=M_2-2H$, and $\mathcal{N}_j=\lfloor h(\rho_j)+\frac12\rfloor$ for $j=1,\ldots,J-1$. Evidently,

$$M_1<\mathcal{N}_0<\mathcal{N}_1<\cdots<\mathcal{N}_{J-1}<\mathcal{N}_J<M_2.$$

Using the above and assuming that t_0 is sufficiently large, we have

$$\begin{aligned}\mathcal{N}_j-h(\alpha_j)=h(\rho_j)-h(\alpha_j)+O(1)&=\frac{t}{2\pi}\frac{\rho_j-\alpha_j}{\alpha_j\rho_j}+O(1)\\ &\asymp\frac{t}{(\frac{t}{M})^2v_jR}\asymp\frac{HR}{v_j}.\end{aligned}$$

By a similar argument we see that $h(\alpha_j)-\mathcal{N}_{j-1}\asymp\frac{HR}{v_j}$, and thus

$$\mathcal{N}_j-\mathcal{N}_{j-1}\asymp\frac{HR}{v_j}. \tag{3.7}$$

Next let

$$\omega_j(x)=\omega(x-\mathcal{N}_{j-1})-\omega(x-\mathcal{N}_j), \tag{3.8}$$

where

$$\omega(x)=\begin{cases}1, & \text{for } x\ge H,\\ \frac12(1+\sin^{s+1}(\frac{\pi x}{2H})), & \text{for } |x|\le H,\\ 0, & \text{for } x\le -H,\end{cases}$$

for an integer $s\ge 2$. These functions provide a C^s partition of unity on the interval $[M_1+2H,M_2-2H]$. In particular,

$$\sum_{j=1}^{J}\omega_j(n)=\begin{cases}0, & \text{for } x\le M_1,\\ 1, & \text{for } M_1+2H\le x\le M_2-2H,\\ 0, & \text{for } x\ge M_2.\end{cases}$$

From this we observe that

$$\left|\sum_{M_1\le n\le M_2}\frac{\lambda_f(n)}{n^{it}}-\sum_{j=1}^{J}\sum_{n=1}^{\infty}\lambda_f(n)n^{-it}\omega_j(n)\right|\le\sum_{m=M_1}^{M_1+2H}|\lambda_f(m)|+\sum_{m=M_2-2H}^{M_2}|\lambda_f(m)|. \tag{3.9}$$
$$\ll_{k,N}\max(H,M^{\frac{3}{5}})(\log M)^{-\delta},$$

by Lemma 2.2(ii). Hence

$$\begin{aligned}\sum_{M_1\le n\le M_2}\frac{\lambda_f(n)}{n^{it}}&=\sum_{j=1}^{J}\sum_{n=1}^{\infty}\lambda_f(n)n^{-it}\omega_j(n)+O_{k,N}(\max(H,M^{\frac{3}{5}})(\log M)^{-\delta})\\&=\sum_{j=1}^{J}\sum_{n=1}^{\infty}\lambda_f(n)e(\alpha_jn)F_j(n)+O_{k,N}(\max(H,M^{\frac{3}{5}})(\log M)^{-\delta}),\end{aligned} \tag{3.10}$$

where $F_j(n)=n^{-it}e(-\alpha_jn)\omega_j(n)$.

We would now want to apply Lemma 2.4 to the sum involving $F_j(n)$. If the level N is not squarefree, then we do not have a Voronoi formula for every Farey fraction α_j. To circumvent this issue, we decompose each fraction into a 'good part' and a 'bad part' part where the bad parts range over a finite set, the additive twists involving the good part of the fraction can be handled using Voronoi summation, and the additive characters involving the bad part can be handled by decomposing into multiplicative characters. To that end, define

$$N^\flat=N\prod_{p|N}p^{-1}\qquad\text{and}\qquad\mathcal{B}(N^\flat)=\left\{\frac{b}{N^\flat}:0\le b<N^\flat\right\}$$

and write

$$\alpha_j=-\frac{u_j}{v_j}=-\frac{a_j}{q_j}+\beta_j=-\frac{a_j}{q_j}+\frac{c_j}{d_j}$$

where $a_j,q_j,c_j,d_j\in\mathbb{Z}_{\ge0}$, $(a_j,q_j)=(c_j,d_j)=(q_j,d_j)=1$, $\beta_j\in\mathcal{B}(N^\flat)$, and for every prime $p\mid q_j$ we have $\mathrm{ord}_p(q_j)\ge\mathrm{ord}_p(N)$. Such a decomposition always exists and is uniquely determined; concretely,

$$d_j=\prod_{\substack{p|v_j\\ \mathrm{ord}_p(v_j)<\mathrm{ord}_p(N)}}p^{\mathrm{ord}_p(v_j)},\quad q_j=\frac{v_j}{d_j},$$

and c_j is the unique integer in $[0,d_j)$ satisfying $q_jc_j\equiv-u_j \pmod{d_j}$. Since $v_j=d_jq_j$, this congruence is equivalent to

$$d_ju_j\equiv-c_jv_j \pmod{d_j^2}.$$

Next we apply Voronoi summation in the form of Lemma 2.4 to see that

(3.11)

$$\sum_{n=1}^{\infty}\lambda_f(n)e(\alpha_jn)F_j(n)=$$

$$\sum_{\beta\in\mathcal{B}(N^\flat)}\sum_{\substack{r|NN^\flat\\(r,q_j)=1}}\sum_{\chi \,(\mathrm{mod}\, N)}c(f,r,\chi;j)\sum_{\ell=1}^{\infty}\lambda_{\bar f\chi}(\ell)e\Big(\frac{\overline{ra_j}\ell}{q_j}\Big)2\pi i^k\int_0^\infty F_j(q_jx)J_{k-1}\Big(4\pi\sqrt{\tfrac{\ell x}{rq_j}}\Big)\,\mathrm{d}x$$

for some complex numbers $c(f,r,\chi;j)$ satisfying $c(f,r,\chi;j) \ll_{N^\flat} 1$. Applying stationary phase to the integral on the right-hand side of this equation we derive the following proposition, deferring the proof until Section 4.

Proposition 3.1. *Given $\beta = \frac{c}{d} \in \mathcal{B}(N^\flat)$ and $r \mid NN^\flat$, let*

$$J(\beta, r) = \{j \in \{1, \ldots, J\} : \beta_j = \beta, (q_j, r) = 1\}.$$

If $s \geq 6$, then

$$\begin{aligned} \sum_{j\in J(\beta,r)} c(f,r,\chi;j) \sum_{\ell=1}^{\infty} \lambda_{\bar f \chi}(\ell) e\Big(\tfrac{\overline{ra_j}\ell}{q_j}\Big) 2\pi i^k \int_0^\infty F_j(q_j x) J_{k-1}\Big(4\pi\sqrt{\tfrac{\ell x}{r q_j}}\Big)\,\mathrm{d}x \\ = \sum_{j\in J(\beta,r)} c(f,r,\chi;j) \sum_{\pm} (\mp 1)^k \sum_{\ell \leq K_1 r d^{-2}} \lambda_{\bar f\chi}(\ell) e\Big(\tfrac{\overline{ra_j}\ell}{q_j}\Big) \omega_j\big(x_j^\pm(\tfrac{\ell}{r})\big) h_j^\pm(\tfrac{\ell}{r}) e\big(g_j^\pm(\tfrac{\ell}{r})\big) \\ + O_{k,N,s}\left(\left(\sqrt{M}\Big(\frac{M}{R^2}\Big)^{\frac{1}{2(s-1)}} + \frac{M^{\frac{5}{2}}R^2}{H^3}\right)(\log M_0)^{-\delta}\right), \end{aligned} \tag{3.12}$$

where $x_j^\pm(\ell)$ are stationary points defined by

$$x_j^\pm(\ell) = \left(\frac{d}{2u_j}\right)^2 \left(\sqrt{\ell + \frac{2tu_jq_j}{\pi d}} \pm \sqrt{\ell}\right)^2, \tag{3.13}$$

$$g_j^\pm(\ell) = -\frac{t}{2\pi}\log x_j^\pm(\ell) + \frac{u_j}{v_j} x_j^\pm(\ell) \mp \frac{2}{q_j}\sqrt{\ell x_j^\pm(\ell)} + \frac{1}{8} \mp \frac{1}{8}, \tag{3.14}$$

$$h_j^\pm(\ell) = \left(\frac{q_j t\sqrt{\ell}}{\pi x_j^\pm(\ell)^{\frac{3}{2}}} \pm \frac{\ell}{x_j^\pm(\ell)}\right)^{-\frac{1}{2}}, \tag{3.15}$$

and $K_1 \asymp M/R^2 = M_0$.

Combining (3.10), (3.11), and Proposition 3.1, we have

$$\begin{aligned} \sum_{M_1 \leq n \leq M_2} \frac{\lambda_f(n)}{n^{it}} = \\ \sum_{\beta\in\mathcal{B}(N^\flat)} \sum_{r \mid NN^\flat} \sum_{\chi \,(\mathrm{mod}\, N)} \sum_{j\in J(\beta,r)} c(f,r,\chi;j) \sum_{\pm}(\mp1)^k \sum_{\ell\leq K_1 r d^{-2}} \lambda_{\bar f\chi}(\ell) e\Big(\tfrac{\overline{ra_j}\ell}{q_j}\Big)\omega_j\big(x_j^\pm(\tfrac{\ell}{r})\big) h_j^\pm(\tfrac{\ell}{r}) e\big(g_j^\pm(\tfrac{\ell}{r})\big) \\ + O_{k,N,s}\left(\left(\sqrt{M}\Big(\frac{M}{R^2}\Big)^{\frac{1}{2(s-1)}} + \frac{M^{\frac{5}{2}}R^2}{H^3} + \max(H, M^{\frac{3}{5}})\right)(\log M_0)^{-\delta}\right). \end{aligned} \tag{3.16}$$

Since $R = \sqrt{M/M_0} \geq 1$, $M \ll \sqrt{N}t$, $M_0 \ll t^{\frac{2}{3}}$, and $|t| \geq N^{\frac{3}{2}}$, we see that

$$\max(H, M^{\frac{3}{5}}) \ll \frac{M^{\frac{5}{2}}R^2}{H^3} \quad \text{and} \quad \frac{M}{R} \ll \frac{R^8 t^3}{M^{\frac{7}{2}}}.$$

Therefore, assuming $s \geq 2$ and using the definition of H, we find that the error term in (3.16) can be replaced by $O_{k,N}(R^8 t^3 M^{-\frac{7}{2}}(\log M_0)^{-\delta})$.

The next step is to split the sum in (3.16) so that the integers ℓ lie in dyadic intervals and the sum over j is reorganized so that the v_j lie in dyadic intervals. Thus we have

(3.17)
$$\sum_{M_1\le n\le M_2}\frac{\lambda_f(n)}{n^{it}}=\sum_{\beta\in\mathcal{B}(N^\flat)}\sum_{r|NN^\flat}\sum_{\chi\,(\mathrm{mod}\,N)}\sum_{\pm}(\mp 1)^k\sum_{L,U,V}$$
$$\cdot\sum_{\ell=L_1}^{L_2}\lambda_{\bar f\chi}(\ell)\sum_{\substack{j\in J(\beta,r)\\ (u_j,v_j)=1\\ U_1\le u_j\le U_2, V_1\le v_j\le V_2\\ du_j\equiv -cv_j\,(\mathrm{mod}\,d^2).}}c(f,r,\chi;j)e(\tfrac{\overline{ra_j}\ell}{q_j})\omega_j(x_j^\pm(\tfrac{\ell}{r}))h_j^\pm(\tfrac{\ell}{r})e(g_j^\pm(\tfrac{\ell}{r}))$$
$$+O_{k,N}(R^8t^3M^{-\frac{7}{2}}(\log M_0)^{-\delta})$$

where L_1,L_2,U_1,U_2,V_1, and V_2 are positive integers satisfying $L\ll L_1\le L_2\le L$, $U\ll U_1\le U_2\le U$, and $V\ll V_1\le V_2\le V$ and L,U,V run through powers of 2 satisfying

$$L\ll\frac{rM}{(dR)^2},\quad V\ll R,\quad\text{and}\quad U\asymp\frac{tV}{M}.$$

Observe that the third condition follows from the fact that if $v_j\asymp V$, then $u_j\asymp\frac{tV}{M}$ since $\frac{u_j}{v_j}\asymp\frac{t}{M}$. The next step is to apply a large sieve inequality for Farey fractions to the inner double sum in (3.17). In fact, we bound a more general sum with Proposition 3.2, below.

3.3. **The large sieve and conclusion of the proof.** To estimate the main term on the right-hand side of (3.16), we employ the following large sieve inequality, deferring the proof until Section 5.

Proposition 3.2. *Let notation be as above and fix $\beta=\frac{c}{d}\in\mathcal{B}(N^\flat)$ and $r\mid NN^\flat$. Let L_1,L_2,U_1,U_2,V_1, and V_2 be positive integers satisfying $L\ll L_1\le L_2\le L$, $U\ll U_1\le U_2\le U$, and $V\ll V_1\le V_2\le V$ where L,U, and V are size parameters satisfying*

$$L\ll\frac{rM}{(dR)^2},\quad V\ll R,\quad\textit{and}\quad U\asymp\frac{tV}{M}.$$

Define

$$\mathcal{R}=\Big\{(u,v)\in\mathbb{Z}^2: U_1\le u\le U_2,\ V_1\le v\le V_2,\ (u,v)=1,\ du\equiv -cv\ (\mathrm{mod}\ d^2)\Big\}$$

and

$$\eta=\sqrt{\frac{d^2t}{rLUV}},\quad X_0=\sqrt{L\max(\eta,1)}.$$

Then, given any complex numbers $\nu(j)$, $\lambda(\ell)$ for $j\in J(\beta,r)$ and $\ell\in\mathbb{Z}\cap[L_1,L_2]$, we have

(3.18)
$$\Bigg|\sum_{\ell=L_1}^{L_2}\lambda(\ell)\sum_{\substack{j\in J(\beta,r)\\ (u_j,v_j)\in\mathcal{R}}}\nu(j)h_j^\pm(\ell/r)\omega_j\big(x_j^\pm(\ell/r)\big)e\big(g_j^\pm(\ell/r)\big)\Bigg|^2$$
$$\ll\max_{j\in J(\beta,r)}\big|\nu(j)\big|^2\sum_{\ell=L_1}^{L_2}|\lambda(\ell)|^2\cdot\frac{\eta rV}{U}\Big\{X_0\big(\#\mathcal{R}\big)^2+\int_{X_0}^{L}B\big(\Delta_1(X),\Delta_2(X)\big)\,\mathrm{d}X\Big\},$$

where the implied constant is absolute, Δ_1 *and* Δ_2 *are functions satisfying*

$$\Delta_1(X) \ll \frac{L}{X^2} \quad \text{and} \quad \Delta_2(X) \ll \frac{L}{\eta X^2}, \tag{3.19}$$

and $B(\Delta_1, \Delta_2)$ *is the number of pairs* $(i,j) \in J(\beta, r)^2$ *such that* $(u_i, v_i), (u_j, v_j) \in \mathcal{R}$,

$$\left\| \frac{\overline{a_i r}}{q_i} - \frac{\overline{a_j r}}{q_j} \right\| \le \Delta_1, \quad \text{and} \quad |u_i v_i - u_j v_j| \le UV\Delta_2. \tag{3.20}$$

For fixed $\beta = \frac{c}{d}$ and r, define $\varphi(y) = \sqrt{y - \beta r}$ and recall that $\frac{u_j}{v_j} = \frac{a_j}{q_j} - \beta$ for $j \in J(\beta, r)$. This, together with the condition $|u_i v_i - u_j v_j| \le UV\Delta_2$, implies that

$$\frac{d}{\sqrt{r}} \left| q_i \varphi\big(ra_i/q_i\big) - q_j \varphi\big(ra_j/q_j\big) \right| = \left| \sqrt{u_i v_i} - \sqrt{u_j v_j} \right| \ll \sqrt{UV}\Delta_2.$$

We want to count pairs of fractions $\left(\frac{ra_i}{q_i}, \frac{ra_j}{q_j}\right)$ in the interval $\left[\frac{rU_1}{V_2} + \beta r, \frac{rU_2}{V_1} + \beta r\right]$ satisfying the inequalities in (3.20). On this interval, we have $\varphi(y) \asymp \Phi := \sqrt{\frac{rU}{V}}$, and hence

$$\left| q_i \varphi\big(ra_i/q_i\big) - q_j \varphi\big(ra_j/q_j\big) \right| \ll \frac{\Phi V}{d} \Delta_2.$$

Note that under the conditions of Proposition 3.2, we have

$$\frac{V_1}{d} \le q_j \le \frac{V_2}{d} \quad \text{and} \quad \frac{r}{d}\left(U_1 + \beta V_1\right) \le ra_j \le \frac{r}{d}\left(U_2 + \beta V_2\right).$$

Assuming that $U_2 \le 2U_1$ and $V_2 \le 2V_1$, we thus see that ra_j and q_j lie in dyadic intervals. This allows us to apply the following estimate of Graham and Kolesnik [10] with parameters $A = \frac{r}{d}\Big(U_1 + \beta V_1\Big)$ and $C = \frac{V_1}{d}$.

Lemma 3.1. *Suppose* A *and* C *are positive integers, and that* Δ_1 *and* Δ_2 *are positive real numbers not exceeding 1. Suppose that* $\varphi(x)$ *is a real positive continuously differentiable function defined on a subinterval* I *of* $[A/(2C), 2A/C]$. *Suppose that there is a constant* C_0 *and a parameter* Φ *such that*

$$\frac{\Phi}{C_0} \le \varphi(x) \le C_0\Phi, \quad \frac{\Phi}{C_0} \le |x\varphi'(x)| \le C_0\Phi, \quad \text{and} \quad \frac{\Phi}{C_0} \le |\varphi(x) - x\varphi'(x)| \le C_0\Phi$$

whenever x *is an element of* I. *Let* B *be the number of solutions of the inequalities*

$$\left\| \frac{\overline{r}}{q} - \frac{\overline{r_1}}{q_1} \right\| \le \Delta_1 \quad \text{and} \quad \Big| q\varphi(r/q) - q_1\varphi(r_1/q_1) \Big| \le C\Phi\Delta_2$$

when $(r,q) = 1, (r_1, q_1) = 1, A < r, r_1 \le 2A,$ *and* $C < q, q_1 \le 2C$ *where* $r\overline{r} \equiv 1 \pmod{q}$ *and* $r_1\overline{r_1} \equiv 1 \pmod{q_1}$. *Then*

$$B \ll \Delta_1\Delta_2 A^2C^2 + \Delta_1^2 A^2 C^2 + AC + \Delta_2 A^2 + \Delta_2 C^2$$

where the implied constant depends only on C_0.

Proof. This is [10, Lemma 7.18] and is a variation of a counting problem first considered by Bombieri and Iwaniec [5] in connection to subconvexity estimates for the Riemann zeta-function. ∎

Since $V \ll R$, $U \ll \frac{tR}{M}$, and $L \ll rM/(dR)^2$, we have $LUV \ll rt/d^2$. Hence

$$\eta \asymp_N \sqrt{\frac{t}{LUV}} \gg_N 1 \quad \text{and} \quad X_0 \asymp_N \sqrt{\eta L} \asymp_N \left(\frac{tL}{UV}\right)^{\frac{1}{4}}. \tag{3.21}$$

Therefore, in the notation of Proposition 3.2, we have

$$\begin{aligned}
&X_0\big(\#\mathcal{R}\big)^2 + \int_{X_0}^{L} B\big(\Delta_1(X), \Delta_2(X)\big)\,\mathrm{d}X \\
&\qquad \ll X_0\big(\#\mathcal{R}\big)^2 + \int_{X_0}^{L} \left\{A^2C^2 + \frac{L^2}{X^4}(1+\eta^{-1}) + AC + \frac{L}{\eta X^2}(A^2+C^2)\right\}\mathrm{d}X \\
&\qquad \ll X_0\big(\#\mathcal{R}\big)^2 + \frac{A^2C^2L^2}{X_0^3}(1+\eta^{-1}) + ACL + \frac{L}{\eta X_0}(A^2+C^2) \\
&\qquad \ll_N X_0U^2V^2 + \frac{U^2V^2L^2}{X_0^3} + LUV + \frac{U^2L}{\eta X_0^2}.
\end{aligned}$$

The inequality $\eta \gg_N 1$ implies that $X_0 \gg_N \sqrt{L}$ which, in turn, implies that the first term on the right-hand side of the above expression dominates the second and fourth terms. Thus, using the estimates in (3.21) and that $U \asymp \frac{tV}{M}$, we have

$$\frac{\eta r V}{U}\left\{X_0\big(\#\mathcal{R}\big)^2 + \int_{X_0}^{L} B\big(\Delta_1(X), \Delta_2(X)\big)\,\mathrm{d}X\right\} \ll_N \left(LMV^2\right)^{\frac{1}{2}} + \left(L^{-1}M^3V^2\right)^{\frac{1}{4}}.$$

We apply the large sieve with $\lambda(\ell) = \lambda_{\bar{f}\chi}(\ell)$ and $\nu(j) = c(f, r, \chi; j)$. By Lemma 2.2(i), we have $\sum_{L_1<\ell\le L_2} |\lambda(\ell)|^2 \ll_{k,N} L$, and thus the left-hand side of the large sieve inequality is $\ll_{k,N} (L^3MV^2)^{\frac{1}{2}} + (L^3M^3V^2)^{\frac{1}{4}}$. Estimating the sums over β, r, and χ trivially in (3.16), we deduce that

$$\sum_{M_1\le n\le M_2} \frac{\lambda_f(n)}{n^{it}} \ll_{k,N} \sum_{L,U,V} \left\{(L^3MV^2)^{\frac{1}{4}} + (L^3M^3V^2)^{\frac{1}{8}}\right\} + \frac{R^8\,t^3}{M^{\frac{7}{2}}}(\log M_0)^{-\delta},$$

where L, U, and V run over powers of 2 satisfying $L \ll rM/(dR)^2$, $V \ll R$, and $U \asymp \frac{tV}{M}$. Note that there are boundedly many values of U corresponding to each V. Hence, we derive that

$$\sum_{M_1\le n\le M_2} \frac{\lambda_f(n)}{n^{it}} \ll_{k,N} \frac{M}{R} + \frac{R^8\,t^3}{M^{\frac{7}{2}}}(\log M_0)^{-\delta} \ll \sqrt{M}\left\{\sqrt{M_0} + M_0^{-4}t^3(\log t)^{-\delta}\right\}.$$

Therefore (3.2) becomes

$$\begin{aligned}
|L(\tfrac{1}{2}+it, f)| &\ll_{k,N} \left\{\sqrt{M_0} + M_0^{-4}t^3(\log t)^{-\delta}\right\}\log C + \sqrt{M_0}(\log M_0)^{-\delta} \\
&\ll_{k,N} \sqrt{M_0}\log t + M_0^{-4}t^3(\log t)^{1-\delta}.
\end{aligned}$$

Choosing $M_0 = \lceil t^{\frac{2}{3}}(\log t)^{-\frac{2\delta}{9}}\rceil$ balances the two terms on the right-hand side and proves Theorem 1.1.

4. Proof of Proposition 3.1

4.1. **Preliminary lemmas.** Before proving Proposition 3.1, we state three lemmas. For k a positive integer, let $C^k([\alpha,\beta])$ denote the space of k times continuously differentiable real-valued functions on the interval $[\alpha,\beta]$. The next two lemmas on exponential integrals are Lemma 5.5.5 and Lemma 5.5.6 of [12].

Lemma 4.1. *Let $F \in C^3([\alpha,\beta])$ and let $G \in C^2([\alpha,\beta])$. Suppose there exist positive parameters M, H, t, U, with $M \geq \beta - \alpha$, and positive constants C_{r_1}, C_{r_2} such that, for $x \in [\alpha,\beta]$, we have*

$$|F^{(r_1)}(x)| \leq C_{r_1} t/M^{r_1} \quad \text{and} \quad |G^{(r_2)}(x)| \leq C_{r_2} U/H^{r_2}$$

for $r_1 \in \{2,3\}$, and $r_2 \in \{0,1,2\}$. If F' and F'' do not change sign on $[\alpha,\beta]$, then

$$\begin{aligned} I = \int_\alpha^\beta G(x) e(F(x))\,\mathrm{d}x &= \frac{G(\beta)e(F(\beta))}{2\pi i F'(\beta)} - \frac{G(\alpha)e(F(\alpha))}{2\pi i F'(\alpha)} \\ &\quad + O\left(\frac{tU}{M^2}\left(1 + \frac{M}{H} + \frac{M^2}{H^2}\frac{\min|F'(x)|}{t/M}\right)\frac{1}{\min|F'(x)|^3}\right). \end{aligned}$$

Lemma 4.2. *Let $F \in C^4([\alpha,\beta])$ and let $G \in C^3([\alpha,\beta])$. Suppose there exist positive parameters M, H, T, U, with $M \geq \beta - \alpha$, $H \geq M/\sqrt{t}$, and positive constants C_{r_1}, C_{r_2} such that, for $x \in [\alpha,\beta]$, we have*

$$|F^{(r_1)}(x)| \leq C_{r_1} t/M^{r_1} \quad \text{and} \quad |G^{(r_2)}(x)| \leq C_{r_2} U/H^{r_2}$$

for $r_1 \in \{2,3,4\}$ and $r_2 \in \{0,1,2,3\}$, and a positive constant $\tilde{C}$ such that

$$F^{(2)}(x) \geq t/\tilde{C}M^2.$$

Suppose also that $F'(x)$ changes sign from negative to positive at a point $x = \gamma$ with $\alpha < \gamma < \beta$. If t is sufficiently large with respect to the constants $C_{r_1}, C_{r_2}, \tilde{C}$, then

$$\begin{aligned} \int_\alpha^\beta G(x)e(F(x))\,\mathrm{d}x &= \frac{G(\gamma)e(F(\gamma)+\frac{1}{8})}{\sqrt{F''(\beta)}} + \frac{G(\beta)e(F(\beta))}{2\pi i F'(\beta)} - \frac{G(\alpha)e(F(\alpha))}{2\pi i F'(\alpha)} \\ &\quad + O\left(\frac{M^4U}{t^2}\left(1+\frac{M}{H}\right)^2\left(\frac{1}{(\gamma-\alpha)^3} + \frac{1}{(\beta-\gamma)^3}\right)\right) + O\left(\frac{MU}{t^{\frac{3}{2}}}\left(1+\frac{M}{H}\right)^2\right). \end{aligned}$$

The third lemma provides bounds for derivatives of F_j.

Lemma 4.3. *For any integer $s \geq 0$, we have*

$$\left|\frac{\mathrm{d}^s}{\mathrm{d}x^s}\left\{F_j(x)\,x^{-\frac{k-1}{2}}\right\}\right| \ll_s (v_jR)^{-s} x^{-\frac{k-1}{2}}.$$

Proof. By continuity, it suffices to consider $x = x_0 \in (\mathcal{N}_{j-1}-H, \mathcal{N}_j+H)\setminus\{\mathcal{N}_{j-1}+H, \mathcal{N}_j-H\}$. For any fixed x_0, the function $F_j(z)z^{-\frac{k-1}{2}}$ agrees with an analytic function $g(z)$ for z in a neighborhood of x_0. We estimate $g^{(s)}(x_0) = \frac{\mathrm{d}^s}{\mathrm{d}x^s}(F_j(x)x^{-\frac{k-1}{2}})|_{x=x_0}$ via the Cauchy integral formula. Since $v_jR \leq R^2 \ll M$, we may fix a small constant $c > 0$ such that $Y = cv_jR \leq \frac{1}{2}M_1$, and integrate over the circle $C_Y(x_0)$ of radius Y around x_0:

$$|g^{(s)}(x_0)| = \left|\frac{s!}{2\pi i}\int_{C_Y(x_0)} \frac{g(z)}{(z-x_0)^{s+1}}\,\mathrm{d}z\right| \leq s!\,Y^{-s} \sup_{z\in C_Y(x_0)} |g(z)| \ll_s (v_jR)^{-s} \sup_{z\in C_Y(x_0)} |g(z)|.$$

Hence, it suffices to show that $g(z) \ll_s x_0^{-\frac{k-1}{2}}$ for $z \in C_Y(x_0)$.

Let $z = x + iy \in C_Y(x_0)$, so that $x = x_0 + O(v_j R) \asymp M$ and $y \ll v_j R$. We have $\frac{z}{x_0} - 1 \ll \frac{v_j R}{x_0} \ll \frac{1}{M_0}$ so, by (3.1) and (3.5), $z^{-\frac{k-1}{2}} = x_0^{-\frac{k-1}{2}} e^{O((k-1)/M_0)} \ll x_0^{-\frac{k-1}{2}}$. Next, observe that

$$|z^{-it} e(-\alpha_j z)| = e^{t \arctan(y/x) + 2\pi\alpha_j y}.$$

The exponent here is bounded since

$$t \arctan\Big(\frac{y}{x}\Big) - \frac{ty}{x} \ll \frac{t(v_j R)^3}{M^3} \ll \frac{t}{M_0^3} \ll 1.$$

Moreover, by the estimates in Section 3.2, we have $x_0 = h(\alpha_j) + O(HR/v_j)$. Since $M_0^2 \gg t$, we also have $v_j R \ll HR/v_j$, so that $x = h(\alpha_j) + O(HR/v_j)$. Thus

$$\frac{ty}{x} + 2\pi\alpha_j y = \frac{2\pi y \alpha_j}{x}(x - h(\alpha_j)) \ll v_j R \cdot \frac{t}{M^2} \cdot \frac{HR}{v_j} \ll 1.$$

Finally, for z near x_0, $\omega_j(z)$ is a linear combination of 1, $\sin^{s+1}(\frac{\pi}{2H}(z - \mathcal{N}_{j-1}))$, and $\sin^{s+1}(\frac{\pi}{2H}(z - \mathcal{N}_j))$. As above, we have $\frac{\pi y}{2H} \ll \frac{v_j R}{H} \ll 1$, so that

$$\sin^{s+1}\Big(\frac{\pi}{2H}(z - \mathcal{N})\Big) \ll_s 1 \quad \text{for } \mathcal{N} \in \{\mathcal{N}_{j-1}, \mathcal{N}_j\},$$

as desired. □

4.2. **Outline of the proof.** Since the proof of Proposition 3.1 is long, we give an outline. The left-hand side of equation (3.12) is written as $\mathcal{S} = \sum_{j \in J(\beta, r)} c(f, r, \chi; j) \mathcal{S}(j)$ where

$$\mathcal{S}(j) = 2\pi i^k \sum_{\ell=1}^{\infty} \lambda_{\bar{f}^\chi}(\ell) e(\tfrac{\overline{r a_j} \ell}{q_j}) I_j(\ell)$$

and

$$I_j(\ell) = \frac{1}{q_j} \int_0^\infty F_j(x) J_{k-1}(\tfrac{4\pi\sqrt{\ell}}{q_j\sqrt{r}}\sqrt{x})\, \mathrm{d}x. \tag{4.1}$$

Our goal is to develop an approximate formula for $\mathcal{S}$. This is done in five steps. In the first four steps we determine an approximate formula for $\mathcal{S}(j)$, and in the final step these approximations are summed over j to obtain our formula for $\mathcal{S}$. We choose real parameters K_1 and K such that $rK_1/d^2 \le K$ and we decompose $\mathcal{S}(j) = \mathcal{S}_{[1,K]}(j) + \mathcal{S}_{(K,\infty)}(j)$, where for an interval $\mathcal{I} \subset \mathbb{R}$, $\mathcal{S}_{\mathcal{I}}(j) := 2\pi i^k \sum_{\ell \in I} \lambda_{\bar{f}^\chi}(\ell) e(\frac{\overline{r a_j}\ell}{q_j}) I_j(\ell)$. Our bounds for the sums $\mathcal{S}_{\mathcal{I}}(j)$ will depend on bounds for $\sum_{\ell \le x} |\lambda_{\bar{f}^\chi}(\ell)|$. These steps are now described in more precise detail.

Step 1. We first bound $\mathcal{S}_{(K,\infty)}(j)$. For $\ell > K$, the integral $I_j(\ell)$ is estimated by integration by parts, making use of the smoothness of F_j and bounds for Bessel functions.

Step 2. Next we insert the asymptotic formula [9, §8.451, Eqn. 1]

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\Big(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\Big) + O\big(x^{-\frac{3}{2}}\big), \tag{4.2}$$

which holds for $\nu \in \mathbb{Z}_{\ge 0}$ as $x \to \infty$, and estimate the corresponding error terms for $\ell \le K$ to deduce that $\mathcal{S}_{[1,K]}(j)$ equals $\tilde{\mathcal{S}}_{[1,K]}(j)$ plus an error term, where $\tilde{\mathcal{S}}_{[1,K]}(j)$ is a simplified sum. We choose K as a function of v_j, M, and R to balance the error terms in steps 1 and 2.

Step 3. We are left with sums of the shape $\tilde{\mathcal{S}}_{[1,K]}(j) = \sum_{\ell \le K} \alpha_{\ell,j,r} \lambda_{\bar{f}\chi}(\ell) e(\frac{\overline{ra_j}\ell}{q_j}) I_j^{\pm}(\ell)$ where $\alpha_{\ell,j,r} \in \mathbb{C}$, $I_j^{\pm}(\ell) = \frac{1}{q_j}\int_0^\infty F_j(x) e(\phi_\pm(x))\,\mathrm{d}x$, and $\phi_\pm(x)$ is a function depending on parameters ℓ, j, t, r. We then choose K_1 so that $\phi'_\pm(x)$ does not change sign for $x \in \mathrm{supp}(F_j)$. For those ℓ with $rK_1 d^{-2} \le \ell \le K$, the integrals $I_j^{\pm}(\ell)$ are estimated using a weighted first derivative estimate (Lemma 4.1).

Step 4. Next, we treat the sum $\sum_{\ell \le rK_1 d^{-2}} \alpha_{\ell,j,r} \lambda_{\bar{f}\chi}(\ell) e(\frac{\overline{ra_j}\ell}{q_j}) I_j^{\pm}(\ell)$. In this range of ℓ the integrals $I_j^{\pm}(\ell)$ possess stationary points $x_j^{\pm}(\frac{\ell}{r})$. Each integral is treated with Lemma 4.2, leading to an expression $\mathcal{S}(j) = \mathcal{M}(j) + O(\widetilde{\mathcal{E}}_1(j) + \widetilde{\mathcal{E}}_2(j) + \widetilde{\mathcal{E}}_3(j))$ where $\mathcal{M}(j)$ is a main term and the $\widetilde{\mathcal{E}}_i(j)$ are error terms.

Step 5. Finally, using $c(f, r, \chi; j) \ll_N 1$, we are left with the sum

$$\mathcal{S} = \sum_{j \in J(\beta, r)} c(f, r, \chi; j)\mathcal{M}(j) + O\Big(\sum_{j=1}^{J} \big(\widetilde{\mathcal{E}}_1(j) + \widetilde{\mathcal{E}}_2(j) + \widetilde{\mathcal{E}}_3(j)\big)\Big).$$

In this last step, the error terms $\widetilde{\mathcal{E}}_i(j)$ are bounded as j ranges over all Farey fractions.

4.3. **Proof of Proposition 3.1.** We now commence with the proof.

Step 1. By repeated application of the identity [9, §8.472, Eqn. 3]

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x),$$

we have

$$\Big(\frac{2}{A}\Big)^s \frac{\mathrm{d}^s}{\mathrm{d}x^s}\Big(x^{\frac{k+s-1}{2}} J_{k+s-1}(A\sqrt{x})\Big) = x^{\frac{k-1}{2}} J_{k-1}(A\sqrt{x}) \text{ for } s \ge 0 \text{ and } A \ne 0. \tag{4.3}$$

Setting $A = \frac{4\pi\sqrt{\ell}}{q_j\sqrt{r}}$, we have $I_j(\ell) = \frac{1}{q_j}\int_0^\infty F_j(x) x^{-\frac{k-1}{2}}(x^{\frac{k-1}{2}} J_{k-1}(A\sqrt{x}))\,\mathrm{d}x$. Integrating by parts s times and using (4.3), it follows that

$$I_j(\ell) = \frac{(-1)^s}{q_j}\Big(\frac{2}{A}\Big)^s \int_0^\infty \frac{\mathrm{d}^s}{\mathrm{d}x^s}(F_j(x)x^{-\frac{k-1}{2}})(x^{\frac{k+s-1}{2}} J_{k+s-1}(A\sqrt{x}))\,\mathrm{d}x.$$

The asymptotic formula in (4.2) gives $J_{k+s-1}(A\sqrt{x}) \ll_{k,s} A^{-\frac{1}{2}} x^{-\frac{1}{4}}$. This estimate, along with Lemma 4.3, implies that

$$|I_j(\ell)| \ll_{k,s} \frac{1}{q_j A^{s+\frac{1}{2}} (v_j R)^s} \int_{\mathrm{supp}(\omega_j)} x^{\frac{2s-1}{4}}\,\mathrm{d}x.$$

Using this bound for $\ell > K$ and $A \asymp \ell^{\frac{1}{2}} q_j^{-1} r^{-\frac{1}{2}}$, we deduce that

$$\mathcal{S}_{(K,\infty)}(j) \ll_{k,s} r^{\frac{s}{2}+\frac{1}{4}} q_j^{s-\frac{1}{2}} M^{\frac{2s-1}{4}} \Big(\frac{1}{v_j R}\Big)^s \sum_{\ell > K} \frac{|\lambda_{\bar{f}\chi}(\ell)|}{\ell^{\frac{s}{2}+\frac{1}{4}}} |\,\mathrm{supp}(\omega_j)|.$$

Note that this last sum only converges for $s \ge 2$. By Lemma 2.2(iii), we have

$$\mathcal{S}_{(K,\infty)}(j) \ll_{k,N,s} r^{\frac{s}{2}+\frac{1}{4}} q_j^{s-\frac{1}{2}} M^{\frac{2s-1}{4}} \Big(\frac{1}{v_j R}\Big)^s |\,\mathrm{supp}(\omega_j)| K^{\frac{3}{4}-\frac{s}{2}} (\log K)^{-\delta}.$$

Step 2. The asymptotic estimate in (4.2) implies that

$$J_{k-1}(\tfrac{4\pi}{q}(\tfrac{\ell x}{r})^{\frac12}) = \frac{r^{\frac14}}{2\sqrt{2}\pi}\frac{q_j^{\frac12}}{(\ell x)^{\frac14}}\Big(e(-\tfrac{k}{4})e(2(\tfrac{\ell x}{rq_j^2})^{\frac12}+\tfrac18)+e(\tfrac{k}{4})e(-2(\tfrac{\ell x}{rq_j^2})^{\frac12}-\tfrac18)\Big)+O_k\Bigg(\frac{r^{\frac34}q_j^{\frac32}}{x^{\frac34}\ell^{\frac34}}\Bigg).$$

Inserting this expression into (4.1) for each $\ell \le K$ and estimating the error terms, we have

$$\mathcal{S}_{[1,K]}(j) = \frac{i^k r^{\frac14}}{\sqrt{2}q_j^{\frac12}}\sum_{\ell\le K}\frac{\lambda_{\bar f\chi}(\ell)e(\frac{\overline{ra_j}\ell}{q_j})}{\ell^{\frac14}}\sum_{\pm}e(\pm\tfrac{k}{4})\int_0^\infty F_j(x)x^{-\frac14}e(\mp2(\tfrac{\ell x}{rq_j^2})^{\frac12}\mp\tfrac18)\,\mathrm{d}x+\mathcal{E}_0(j),$$

where

$$\mathcal{E}_0(j) \ll \sum_{\ell\le K}\frac{|\lambda_{\bar f\chi}(\ell)|}{\ell^{\frac34}}\frac{1}{q_j}\int_0^\infty |F_j(x)|\frac{r^{\frac34}q_j^{\frac32}}{x^{\frac34}}\,\mathrm{d}x \ll_{k,N} \frac{r^{\frac34}q_j^{\frac12}}{M^{\frac34}}|\operatorname{supp}(\omega_j)|K^{\frac14}(\log K)^{-\delta}$$

by Lemma 2.2(iv) since $x\asymp M$. Choosing $K := (\frac{M}{v_jR})^{\frac{2}{s-1}}M_0$ and recalling that $|\operatorname{supp}(\omega_j)| \ll \frac{HR}{v_j}$, it follows that

$$\mathcal{E}_1(j) := \mathcal{S}_{(K,\infty)}(j)+\mathcal{E}_0(j) \ll_{k,N,s} \frac{HM^{\frac{2-s}{2(s-1)}}R^{\frac{s-2}{2(s-1)}}}{v_j^{\frac{s}{2(s-1)}}}(\log M_0)^{-\delta}. \tag{4.4}$$

Therefore $\mathcal{S}(j) = \tilde{\mathcal{S}}_{[1,K]}(j)+\mathcal{E}_1(j)$, where

$$\tilde{\mathcal{S}}_{[1,K]}(j) = i^k\sum_{\ell\le K}\frac{\lambda_{\bar f\chi}(\ell)e(\frac{\overline{ra_j}\ell}{q_j})}{\sqrt{2(\frac{\ell}{r})^{\frac12}q_j}}(e(\tfrac{k}{4})I_j^+(\ell)+e(-\tfrac{k}{4})I_j^-(\ell)), \tag{4.5}$$

$$I_j^{\pm}(\ell) = \int_0^\infty g_j(x)e(\phi_\pm(x))\,\mathrm{d}x, \tag{4.6}$$

$$g_j(x) = x^{-\frac14}\omega_j(x), \tag{4.7}$$

and

$$\phi_\pm(x) := -\frac{t}{2\pi}\log x-\alpha_j x\mp 2(\tfrac{\ell x}{rq_j^2})^{\frac12}\mp\tfrac18. \tag{4.8}$$

Step 3. Let $K_1 = \lceil\frac{CM}{R^2}\rceil$, where C is a sufficiently large positive constant. For ℓ satisfying $rK_1d^{-2}\le \ell\le K$, we bound the integral in (4.6) by showing that $|\phi_\pm'(x)| \gg \frac{1}{q_j}(\frac{\ell}{rM})^{\frac12}$ and using a weighted first derivative estimate (Lemma 4.1). It is convenient to write

$$\phi_+(x) = \mathbf{f}(x)+\frac{u_j}{v_j}x-\frac{2}{q_j}\Big(\frac{\ell x}{r}\Big)^{\frac12}-\frac18,$$

where $\mathbf{f}(x) = -\frac{t}{2\pi}\log x$. We shall make use of

$$\mathbf{f}^{(j)}(x) = \frac{(j-1)!(-1)^j t}{2\pi x^j} \asymp \frac{t}{M^j}\ \text{for } x\in[M_1,M_2] \tag{4.9}$$

for $j\ge1$ and the identity

$$y = \mathbf{f}'(h(y)). \tag{4.10}$$

For $x \in \operatorname{supp}(\omega_j)$, the mean value theorem implies there exists $\xi \in \operatorname{supp}(\omega_j)$ such that

$$\mathbf{f}'(x) = \mathbf{f}'(h(\rho_j)) + \mathbf{f}''(\xi)(x - h(\rho_j)) = \rho_j + O\Big(\frac{t}{M^2}|\operatorname{supp}(\omega_j)|\Big),$$

by (4.9) and (4.10). By (3.6) and (3.7) this is

$$\mathbf{f}'(x) = \alpha_j + O\Big(\frac{1}{v_j R} + \frac{t}{M^2}\frac{HR}{v_j}\Big).$$

Let c_2 be such that $M_2 \le c_2 M$. By the previous equation there exists $c_0 > 0$ such that

$$|\mathbf{f}'(x) - \alpha_j| \le c_0\Big(\frac{1}{v_j R} + \frac{t}{M^2}\frac{HR}{v_j}\Big) \le \frac{2c_0}{q_j dR} \le \frac{1}{2q_j}\Big(\frac{\ell}{rc_2M}\Big)^{\frac{1}{2}}, \tag{4.11}$$

as long as $\frac{\ell}{r} \gg \frac{M}{d^2R^2}$. Since $x \le M_2 \le c_2 M$, we obtain $|\mathbf{f}'(x) - \alpha_j| \le \frac{1}{2q_j}(\frac{\ell}{rx})^{\frac{1}{2}}$. It follows from (4.8) and (4.11) that for $\frac{\ell}{r} \ge K_1 d^{-2}$,

$$|\phi_+'(x)| = |\mathbf{f}'(x) - \alpha_j - \tfrac{1}{q_j}(\tfrac{\ell}{rx})^{\frac{1}{2}}| \ge \tfrac{1}{2q_j}(\tfrac{\ell}{rx})^{\frac{1}{2}} \gg \tfrac{1}{2q_j}(\tfrac{\ell}{rM})^{\frac{1}{2}}.$$

We now compute the derivatives of $F(x) = \phi_+(x)$ and $G(x) = g_j(x)$ given by (4.7). We have

$$F^{(r_1)}(x) = \mathbf{f}^{(r_1)}(x) - \frac{2}{q_j}\sqrt{\frac{\ell}{r}}\frac{\mathrm{d}^{r_1}}{\mathrm{d}x^{r_1}}(x^{\frac{1}{2}}) \ll \frac{t}{M^{r_1}} + \frac{2}{q_j}\sqrt{\frac{\ell}{r}}\frac{1}{M^{r_1-\frac{1}{2}}} \ll \frac{t}{M^{r_1}}, \tag{4.12}$$

which follows (after some calculation) using the facts that $\ell \le K$ and $s \ge 6$. Also we have

$$G^{(r_2)}(x) = \sum_{i_1+i_2=r_2}\binom{r_2}{i_1}\frac{\mathrm{d}^{i_1}}{\mathrm{d}x^{i_1}}x^{-\frac{1}{4}}\omega_j^{(i_2)}(x) \ll \sum_{i_1+i_2=r_2} M^{-\frac{1}{4}-i_1}H^{-i_2} \ll M^{-\frac{1}{4}}H^{-r_2} \tag{4.13}$$

where we used $H = \frac{M^2}{R^2t} \le M$. We now invoke Lemma 4.1 with $\alpha = \mathcal{N}_{j-1} - H$, $\beta = \mathcal{N}_j + H$, and $U = M^{-\frac{1}{4}}$ and make use of the lower bound $F'(x) \gg \frac{1}{q_j}(\frac{\ell}{rM})^{\frac{1}{2}}$ for $x \in [\alpha, \beta]$. With these choices the condition $M \ge \beta - \alpha$ is satisfied and $G(\alpha) = G(\beta) = 0$. Thus, for $rK_1d^{-2} \le \ell \le K$, this lemma gives

$$I_j^{\pm}(\ell) \ll M^{-\frac{1}{4}}\frac{t}{M^2}\Big(1 + \frac{M}{H} + \frac{M^2}{H^2}\frac{\frac{1}{q_j}(\frac{\ell}{rM})^{\frac{1}{2}}}{t/M}\Big)\Big(\frac{q_j^2 rM}{\ell}\Big)^{\frac{3}{2}}.$$

Since $H \le M$, we also have

$$\begin{aligned} r^{\frac{1}{4}}q_j^{-\frac{1}{2}}\ell^{-\frac{1}{4}}I_j^{\pm}(\ell) &\ll r^{\frac{1}{4}}(q_j^2\ell M)^{-\frac{1}{4}}\frac{t}{M^2}\Big(\frac{M}{H} + \frac{M^3}{H^2t}\frac{1}{q_j}\Big(\frac{\ell}{rM}\Big)^{\frac{1}{2}}\Big)\frac{q_j^3 r^{\frac{3}{2}}M^{\frac{3}{2}}}{\ell^{\frac{3}{2}}} \\ &\ll \frac{r^{\frac{7}{4}}}{\ell^{\frac{1}{2}}}\Big(q_j^{\frac{5}{2}}\Big(\frac{M}{\ell}\Big)^{\frac{5}{4}}\frac{M}{H^2R^2} + q_j^{\frac{3}{2}}\Big(\frac{M}{\ell}\Big)^{\frac{3}{4}}\frac{M}{H^2\sqrt{r}}\Big), \end{aligned}$$

where we used the definition of H in the last line. It follows that the contribution of the range $\ell \in (rK_1d^{-2}, K]$ to (4.5) is

$$\begin{aligned}\mathcal{E}_2(j) &\ll r^{\frac{7}{4}} \sum_{rK_1d^{-2}<\ell\leq K} \frac{|\lambda_{\bar{f}\chi}(\ell)|}{\ell^{\frac{1}{2}}}\Big(q_j^{\frac{5}{2}}\Big(\frac{M}{\ell}\Big)^{\frac{5}{4}}\frac{M}{H^2R^2} + q_j^{\frac{3}{2}}\Big(\frac{M}{\ell}\Big)^{\frac{3}{4}}\frac{M}{H^2\sqrt{r}}\Big)\\ &\ll r^{\frac{7}{4}}\Big(\frac{q_j^{\frac{5}{2}}M^{\frac{9}{4}}}{H^2R^2}\sum_{\ell>rK_1d^{-2}}\frac{|\lambda_{\bar{f}\chi}(\ell)|}{\ell^{\frac{7}{4}}} + \frac{q_j^{\frac{3}{2}}M^{\frac{7}{4}}}{H^2\sqrt{r}}\sum_{\ell>rK_1d^{-2}}\frac{|\lambda_{\bar{f}\chi}(\ell)|}{\ell^{\frac{5}{4}}}\Big).\end{aligned}$$

Using Lemma 2.2(iii) and the estimate $K_1 \asymp M/R^2$, we have

$$\begin{aligned}\mathcal{E}_2(j) &\ll_{k,N} r^{\frac{7}{4}}\Big(\frac{q_j^{\frac{5}{2}}M^{\frac{9}{4}}}{H^2R^2}\Big(\frac{rM}{R^2d^2}\Big)^{-\frac{3}{4}} + \frac{q_j^{\frac{3}{2}}M^{\frac{7}{4}}}{H^2\sqrt{r}}\Big(\frac{rM}{R^2d^2}\Big)^{-\frac{1}{4}}\Big)(\log(2+rK_1d^{-2}))^{-\delta}\\ &\ll_{k,N} \frac{v_j^{\frac{3}{2}}M^{\frac{3}{2}}}{H^2}\Big(\frac{v_j}{R^{\frac{1}{2}}} + R^{\frac{1}{2}}\Big)(\log M_0)^{-\delta} \ll \Big(\frac{v_j}{R}\Big)^{\frac{3}{2}}\frac{M^{\frac{3}{2}}R^2}{H^2}(\log M_0)^{-\delta},\end{aligned} \tag{4.14}$$

since $v_j \leq R$.

Step 4. We have shown that $\mathcal{S}(j) = \tilde{\mathcal{S}}_{[1,rK_1d^{-2}]}(j) + \mathcal{E}_1(j) + \mathcal{E}_2(j)$ where

$$\tilde{\mathcal{S}}_{[1,rK_1d^{-2}]}(j) = i^k \sum_{\ell\leq rK_1d^{-2}} \frac{\lambda_{\bar{f}\chi}(\ell)e(\frac{\overline{ra_j}\ell}{q_j})}{\sqrt{2(\frac{\ell}{r})^{\frac{1}{2}}q_j}}(e(\tfrac{k}{4})I_j^+(\ell) + e(-\tfrac{k}{4})I_j^-(\ell)), \tag{4.15}$$

and $\mathcal{E}_1(j)$ and $\mathcal{E}_2(j)$ are estimated by (4.4) and (4.14), respectively. For $\ell \leq rK_1d^{-2}$, we extract the stationary phase terms of the integrals $I_j^{\pm}(\ell)$ given by (4.6). Let $x_j^{\pm}(\ell)$ be the roots of

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(-\frac{t}{2\pi}\log x - \alpha_j x \mp 2(\tfrac{lx}{q_j^2})^{\frac{1}{2}} \mp \tfrac{1}{8}\Big) = 0.$$

Notice that the numbers $x_j^{\pm}(\frac{\ell}{r})$ are the stationary points satisfying $\phi'_{\pm}(x_j^{\pm}(\frac{\ell}{r})) = 0$. It follows that the $x_j^{\pm}(\ell)$ are the positive roots of

$$-\frac{t}{2\pi x} + \frac{u_j}{v_j} \mp \Big(\frac{\ell}{x}\Big)^{\frac{1}{2}}\frac{1}{q_j} = 0, \tag{4.16}$$

since $\alpha_j = -\frac{u_j}{v_j}$. Multiplying by x this becomes $\frac{u_j}{v_j}x \mp \frac{\ell^{\frac{1}{2}}}{q_j}\sqrt{x} - \frac{t}{2\pi} = 0$ so that

$$\sqrt{x} = \frac{\pm\frac{\ell^{\frac{1}{2}}}{q_j} \pm \sqrt{\frac{\ell}{q_j^2} + 4\frac{u_j}{v_j}\frac{t}{2\pi}}}{2\frac{u_j}{v_j}}.$$

We discard those solutions corresponding to the second $-$ sign since $\sqrt{x}$ is necessarily positive. With a little calculation, it follows that

$$x_j^{\pm}(\ell) = \left(\frac{\pm\frac{\ell^{\frac{1}{2}}}{q_j} + \sqrt{\frac{\ell}{q_j^2} + \frac{2u_jq_j^2t/\pi v_j}{q_j^2}}}{2\frac{u_j}{v_j}}\right)^2 = \Big(\frac{d}{2u_j}\Big)^2\Big(\sqrt{\ell + \frac{2u_jq_jt}{\pi d}} \pm \sqrt{\ell}\Big)^2,$$

since $v_j = dq_j$. Finally, we apply the stationary phase lemma (Lemma 4.2) to those $I_j^+(\ell)$ with $\ell \le rK_1 d^{-2}$. We choose $F(x) = \phi_+(x)$, $G(x) = g_j(x)$, $\alpha = x_j^+(\frac{\ell}{r}) - \frac{M}{4}$, $\beta = x_j^+(\frac{\ell}{r}) + \frac{M}{4}$, and $\gamma = x_j^+(\frac{\ell}{r})$. We also have the parameters t, M, H, and $U = M^{-\frac{1}{4}}$ which correspond to those of Lemma 4.2 and we have the derivative bounds (4.12) and (4.13). Observe that the conditions $M \ge \beta - \alpha$ and $H \ge M/\sqrt{t}$ are both met. With these choices we now demonstrate that $\mathrm{supp}(\omega_j) = [\mathcal{N}_{j-1} - H, \mathcal{N}_j + H] \subseteq [\alpha, \beta]$ so that $G(\alpha) = G(\beta) = 0$. We aim to show $\alpha = x_j^+(\frac{\ell}{r}) - \frac{M}{4} \le \mathcal{N}_{j-1} - H$. By the mean value theorem there exists $\xi \in \mathrm{supp}(\omega_j)$ such that

$$|\mathbf{f}'(\mathcal{N}_{j-1}) - \mathbf{f}'(x_j^+(\tfrac{\ell}{r}))| = |\mathbf{f}''(\xi)||\mathcal{N}_{j-1} - x_j^+(\tfrac{\ell}{r})| \gg \frac{t}{M^2}|\mathcal{N}_{j-1} - x_j^+(\tfrac{\ell}{r})|. \tag{4.17}$$

Similarly there exists $\xi' \in \mathrm{supp}(\omega_j)$ such that

$$\mathbf{f}'(\mathcal{N}_{j-1}) = \mathbf{f}'(h(\rho_{j-1})) + O(\mathbf{f}''(\xi')) = \rho_{j-1} + O\Big(\frac{t}{M^2}\Big),$$

by (4.10). By (4.16) $\mathbf{f}'(x_j^+(\frac{\ell}{r})) = -\frac{t}{2\pi x_j^+(\frac{\ell}{r})} = \alpha_j \pm \left(\frac{\ell/r}{x_j^+(\frac{\ell}{r})}\right)^{\frac{1}{2}} \frac{1}{q_j}$. Using (3.6) it follows that

$$|\mathbf{f}'(\mathcal{N}_{j-1}) - \mathbf{f}'(x_j^+(\tfrac{\ell}{r}))| \ll \frac{1}{v_j R} + \frac{t}{M^2} + \Big(\frac{\ell/r}{M}\Big)^{\frac{1}{2}}\frac{1}{q_j} \ll \frac{1}{v_j R} + \frac{t}{M^2}, \tag{4.18}$$

since $\frac{\ell}{r} \le \frac{CM}{d^2R^2}$. Combining (4.17) and (4.18) yields $|\mathcal{N}_{j-1} - x_j^+(\frac{\ell}{r})| \ll \frac{M^2}{tR} + 1$ and thus

$$x_j^+(\tfrac{\ell}{r}) - \mathcal{N}_{j-1} + H \le O\Big(\frac{M^2}{tR} + 1\Big) + \frac{M^2}{R^2 t} \le \frac{M}{4},$$

assuming that t_0 is sufficiently large. Hence $\alpha \le \mathcal{N}_{j-1} - H$, and an analogous argument establishes that $\mathcal{N}_j + H \le \beta$. The stationary point of $G = \phi_+$ is $x_j^+(\frac{\ell}{r})$. Hence the main term in Lemma 4.2 is

$$\frac{x_j^+(\frac{\ell}{r})^{-\frac{1}{4}}\omega_j(x_j^+(\frac{\ell}{r}))e(\phi_+(x_j^+(\frac{\ell}{r})) + \frac{1}{8})}{\sqrt{\phi_+''(x_j^+(\frac{\ell}{r}))}},$$

and since $\gamma - \alpha = \beta - \gamma = \frac{M}{4}$, the error term is

$$\ll \frac{M^4 M^{-\frac{1}{4}}}{t^2}\Big(1 + \frac{M}{H}\Big)^2 M^{-3} + \frac{M M^{-\frac{1}{4}}}{t^{\frac{3}{2}}}\Big(1 + \frac{M}{H}\Big)^2 \ll M^{-\frac{1}{4}}\frac{M^3}{t^{\frac{3}{2}}H^2},$$

as the second error term dominates the first and $H \le M$. A similar argument establishes the analogous result for $I_j^-(\ell)$. Thus,

$$I_j^\pm(\ell) = \frac{x_j^\pm(\frac{\ell}{r})^{-\frac{1}{4}}\omega_j(x_j^\pm(\frac{\ell}{r}))e(\phi_\pm(x_j^\pm(\frac{\ell}{r})) + \frac{1}{8})}{\sqrt{\phi_\pm''(x_j^\pm(\frac{\ell}{r}))}} + O\Big(\frac{M^{\frac{11}{4}}}{t^{\frac{3}{2}}H^2}\Big) \text{ for } \ell \le rK_1 d^{-2}.$$

The error term, when inserted into (4.15), becomes

$$\mathcal{E}_3(j) \ll \frac{r^{\frac{1}{4}}}{q_j^{\frac{1}{2}}} \sum_{\ell \le rK_1 d^{-2}} \frac{|\lambda_{\bar{f}\chi}(\ell)| M^{\frac{11}{4}}}{\ell^{\frac{1}{4}} H^2 t^{\frac{3}{2}}} \ll_{k,N} \frac{r^{\frac{1}{4}} M^{\frac{11}{4}}}{q_j^{\frac{1}{2}} H^2 t^{\frac{3}{2}}}\Big(\frac{rM}{R^2 d^2}\Big)^{\frac{3}{4}} (\log M_0)^{-\delta},$$

by Lemma 2.2(iv) and using $K_1 \asymp \frac{M}{R^2}$. It follows that

$$\mathcal{E}_3(j) \ll_{k,N} \frac{M^{\frac{7}{2}}(\log M_0)^{-\delta}}{q_j^{\frac{1}{2}} H^2 t^{\frac{3}{2}} R^{\frac{3}{2}}} \ll_{k,N} \frac{M^{\frac{1}{2}} R^{\frac{3}{2}}}{\sqrt{H v_j}} (\log M_0)^{-\delta},$$

since $H \asymp \frac{M^2}{R^2 t}$ and $v_j = dq_j$. Hence we have established

$$\mathcal{S}(j) = \mathcal{M}(j) + O_{k,N,s}\Big(\big(\widetilde{\mathcal{E}}_1(j) + \widetilde{\mathcal{E}}_2(j) + \widetilde{\mathcal{E}}_3(j)\big)(\log M_0)^{-\delta}\Big),$$

where

$$\begin{aligned}\mathcal{M}(j) =& i^k e(\tfrac{k}{4}) \sum_{\ell \le rK_1 d^{-2}} \frac{\lambda_{\bar{f}\chi}(l) e(\frac{\overline{ra_j} l}{q_j}) \omega_j(x_j^+(\frac{l}{r})) e(\phi_+(x_j^+(\frac{l}{r})) + \frac{1}{8})}{\sqrt{2(\frac{\ell}{r})^{\frac{1}{2}} q_j} x_j^+(\frac{l}{r})^{\frac{1}{4}} \sqrt{\phi_+''(x_j^+(\frac{l}{r}))}} \\ &+ i^k e(-\tfrac{k}{4}) \sum_{\ell \le rK_1 d^{-2}} \frac{\lambda_{\bar{f}\chi}(l) e(\frac{\overline{ra_j} l}{q_j}) \omega_j(x_j^-(\frac{l}{r})) e(\phi_-(x_j^-(\frac{l}{r})) + \frac{1}{8})}{\sqrt{2(\frac{\ell}{r})^{\frac{1}{2}} q_j} x_j^-(\frac{l}{r})^{\frac{1}{4}} \sqrt{\phi_-''(x_j^-(\frac{l}{r}))}},\end{aligned} \tag{4.19}$$

$\widetilde{\mathcal{E}}_1(j) = \frac{HM^{\frac{2-s}{2(s-1)}} R^{\frac{s-2}{2(s-1)}}}{v_j^{\frac{s}{2(s-1)}}}$, $\widetilde{\mathcal{E}}_2(j) = (\frac{v_j}{R})^{\frac{3}{2}} \frac{M^{\frac{3}{2}} R^2}{H^2}$, and $\widetilde{\mathcal{E}}_3(j) = \frac{M^{\frac{1}{2}} R^{\frac{3}{2}}}{\sqrt{Hv_j}}$. We now simplify the expression for $\mathcal{M}(j)$. By (4.8), it follows that

$$x^{\frac{1}{4}}(\phi_\pm''(x))^{\frac{1}{2}} = (x^{\frac{1}{2}})^{\frac{1}{2}}\Big(\frac{t}{2\pi x^2} \pm \frac{1}{2q_j}\sqrt{\frac{\ell}{r}} x^{-\frac{3}{2}}\Big)^{\frac{1}{2}} = \Big(\frac{t}{2\pi x^{\frac{3}{2}}} \pm \frac{1}{2q_j}\sqrt{\frac{\ell}{r}} x^{-1}\Big)^{\frac{1}{2}},$$

and thus $\sqrt{2(\frac{\ell}{r})^{\frac{1}{2}} q_j} x_j^\pm(\frac{\ell}{r})^{\frac{1}{4}} (\phi_\pm''(x_j^\pm(\frac{\ell}{r})))^{\frac{1}{2}} = \Big(\frac{\sqrt{\frac{\ell}{r}} q_j t}{\pi x_j^\pm(\frac{\ell}{r})^{\frac{3}{2}}} \pm \frac{\ell}{r} \frac{1}{x_j^\pm(\frac{\ell}{r})}\Big)^{\frac{1}{2}}$. Since $i^k e(\pm\frac{k}{4}) = (\mp 1)^k$, the expression in (4.19) simplifies to

$$\mathcal{M}(j) = \sum_{\pm} (\mp 1)^k \sum_{\ell=1}^{rK_1 d^{-2}} \lambda_{\bar{f}\chi}(\ell) e(\tfrac{\overline{ra_j}\ell}{q_j}) \omega_j(x_j^\pm(\tfrac{\ell}{r})) h_j^\pm(\tfrac{\ell}{r}) e(g_j^\pm(\tfrac{\ell}{r})), \tag{4.20}$$

where $x_j^\pm(\ell)$, $h_j^\pm(\ell)$, and $g_j^\pm(\ell)$ are given by (3.13), (3.14), and (3.15), respectively. Therefore,

$$\mathcal{S} = \sum_{j \in J(\beta,r)} c(f,r,\chi;j)\mathcal{M}(j) + O_{k,N,s}\Big((\log M_0)^{-\delta} \sum_{j=1}^{J} \big(\widetilde{\mathcal{E}}_1(j) + \widetilde{\mathcal{E}}_2(j) + \widetilde{\mathcal{E}}_3(j)\big)\Big), \tag{4.21}$$

since $\sum_{j \in J(\beta,r)} \widetilde{\mathcal{E}}_i(j) \le \sum_{j=1}^{J} \widetilde{\mathcal{E}}_i(j)$.

Step 5. In this final step, we bound the error terms in (4.21). First, we divide the sum over j into subsums where the v_j lie in dyadic intervals $[Q, 2Q]$ where $Q = 2^i$, $i \ge 0$, and $Q \le R$. We require a bound for the number of v_j in $[Q, 2Q]$. Observe that the Farey fractions $-\frac{u_j}{v_j}$ lie in the interval $\mathcal{I} = [-\frac{t}{2\pi(M_1+2H)}, -\frac{t}{2\pi(M_2-2H)}]$ of length $|\mathcal{I}| \asymp \frac{t}{M} \asymp \frac{M}{HR^2}$, since $H \asymp \frac{M^2}{R^2 t}$. Now if $\mathcal{F}(Q)$ denotes the extended Farey fractions with denominator less than or equal to Q, then [12, Lemma 1.2.3] gives

$$\sum_{\alpha \in \mathcal{F}(Q) \cap I} 1 \le \Delta Q^2 + 1,$$

where I is an interval of length Δ. Applying this estimate with $\Delta = \frac{M}{HR^2}$, we have

$$\sum_{Q\le v_j\le 2Q} 1 \le \sum_{\alpha\in\mathcal{F}(2Q)\cap\mathcal{I}} 1 \ll \frac{MQ^2}{HR^2}+1. \tag{4.22}$$

Consequently,

$$\begin{aligned}
\sum_{j=1}^{J}\widetilde{\mathcal{E}}_1(j) &\ll H\sum_{Q\le R} M^{\frac{2-s}{2(s-1)}}R^{\frac{s-2}{2(s-1)}}\Big(\sum_{Q\le v_j\le 2Q} v_j^{-\frac{s}{2(s-1)}}\Big)\\
&\ll H\sum_{Q\le R}\frac{M^{\frac{2-s}{2(s-1)}}R^{\frac{s-2}{2(s-1)}}}{Q^{\frac{s}{2(s-1)}}}\Big(\frac{MQ^2}{HR^2}+1\Big)\\
&= H\sqrt{\frac{R}{M}}\Big(\frac{M}{R}\Big)^{\frac{1}{2(s-1)}}\Big(\frac{M}{HR^2}\sum_{Q\le R}Q^{\frac{3}{2}-\frac{1}{2(s-1)}}+\sum_{Q\le R}Q^{-\frac{1}{2}-\frac{1}{2(s-1)}}\Big),
\end{aligned}$$

Using the elementary estimate

$$\sum_{\substack{Q\le R\\ Q=2^i, i\ge 0}} Q^{c_1} \ll_{c_1} \begin{cases} R^{c_1}, & \text{for } c_1>0,\\ 1, & \text{for } c_1<0,\end{cases} \tag{4.23}$$

with $c_1 = \frac{3}{2}-\frac{s}{2(s-1)}$ and $c_1 = -\frac{1}{2}-\frac{s}{2(s-1)}$ (for $s\ge 2$) it follows that

$$\begin{aligned}
\sum_{j=1}^{J}\widetilde{\mathcal{E}}_1(j) &\ll H\sqrt{\frac{R}{M}}\Big(\frac{M}{R}\Big)^{\frac{1}{2(s-1)}}\Big(\frac{M}{HR^2}R^{\frac{3}{2}-\frac{1}{2(s-1)}}+1\Big)\\
&\ll \sqrt{\frac{R}{M}}\Big(\frac{M}{R}\Big)^{\frac{1}{2(s-1)}}\Big(\frac{M}{R^2}R^{\frac{3}{2}-\frac{1}{2(s-1)}}+H\Big).
\end{aligned}$$

Observe that $H \ll \frac{M^2}{R^2 t} \ll \frac{M}{R}$. Since $s\ge 6$, the second term in the brackets is bounded by the first, and

$$\sum_{j=1}^{J}\widetilde{\mathcal{E}}_1(j) \ll \sqrt{\frac{R}{M}}\Big(\frac{M}{R}\Big)^{\frac{1}{2(s-1)}}\frac{M}{R^2}R^{\frac{3}{2}-\frac{1}{2(s-1)}} = \sqrt{M}\Big(\frac{M}{R^2}\Big)^{\frac{1}{2(s-1)}}. \tag{4.24}$$

Turning to the second error term in (4.21), we have

$$\sum_{j=1}^{J}\widetilde{\mathcal{E}}_2(j) \ll \frac{M^{\frac{3}{2}}R^2}{H^2}\sum_{Q\le R}\sum_{Q\le v_j\le 2Q}\Big(\frac{v_j}{R}\Big)^{\frac{3}{2}} \ll \frac{M^{\frac{3}{2}}R^2}{H^2}\sum_{Q\le R}\Big(\frac{Q}{R}\Big)^{\frac{3}{2}}\Big(\frac{MQ^2}{HR^2}+1\Big).$$

Again applying (4.23) with $c_1=\frac{7}{2}$ and $c_1=\frac{3}{2}$, we find that

$$\sum_{j=1}^{J}\widetilde{\mathcal{E}}_2(j) \ll \frac{M^{\frac{3}{2}}R^2}{H^2}\Big(\frac{M}{H}\sum_{Q\le R}\Big(\frac{Q}{R}\Big)^{\frac{7}{2}}+\sum_{Q\le R}\Big(\frac{Q}{R}\Big)^{\frac{3}{2}}\Big) \ll \frac{M^{\frac{3}{2}}R^2}{H^2}\Big(\frac{M}{H}+1\Big) \ll \frac{M^{\frac{5}{2}}R^2}{H^3}, \tag{4.25}$$

since $H \ll \frac{M}{R} \ll M$. The third error term in (4.21) is

$$\sum_{j=1}^{J} \widetilde{\mathcal{E}}_3(j) \ll \frac{M^{\frac12}R^{\frac32}}{\sqrt{H}} \sum_{Q\le R}\sum_{Q\le v_j\le 2Q} v_j^{-\frac12} \ll \frac{M^{\frac12}R^{\frac32}}{\sqrt{H}} \sum_{Q\le R} Q^{-\frac12}\Big(\frac{MQ^2}{HR^2}+1\Big),$$

by (4.22). By (4.23) with $c_1 = \frac32$ and $c_2 = -\frac12$, we have

$$\sum_{j=1}^{J} \widetilde{\mathcal{E}}_3(j) \ll \frac{M^{\frac12}R^{\frac32}}{\sqrt{H}}\Big(\frac{M}{HR^2}R^{\frac32}+1\Big) = \frac{M^{\frac32}R}{H^{\frac32}}\Big(1+\frac{R^{\frac12}H}{M}\Big) \ll \frac{M^{\frac32}R}{H^{\frac32}}, \tag{4.26}$$

since $\frac{R^{\frac12}H}{M} \ll R^{-\frac12} \ll 1$. Collecting the estimates in (4.24), (4.25), and (4.26), we find that

$$\sum_{j=1}^{J}\big(\widetilde{\mathcal{E}}_1(j)+\widetilde{\mathcal{E}}_2(j)+\widetilde{\mathcal{E}}_3(j)\big) \ll \sqrt{M}\Big(\frac{M}{R^2}\Big)^{\frac{1}{2(s-1)}} + \frac{M^{\frac52}R^2}{H^3} + \frac{M^{\frac32}R}{H^{\frac32}}. \tag{4.27}$$

Note that the third error term is dominated by the first two. To see this, note that if $MR \ge H^{\frac32}$, then $\frac{M^{\frac32}R}{H^{\frac32}} \le \frac{M^{\frac52}R^2}{H^3}$ while if $MR \le H^{\frac32}$, then $\frac{M^{\frac32}R}{H^{\frac32}} = \sqrt{M}\frac{MR}{H^{\frac32}} \le \sqrt{M} \ll \sqrt{M}(\frac{M}{R^2})^{\frac{1}{2(s-1)}}$. Therefore the right-hand side of (4.27) is $O((\sqrt{M}(\frac{M}{R^2})^{\frac{1}{2(s-1)}} + \frac{M^{\frac52}R^2}{H^3})(\log M_0)^{-\delta})$. Proposition 3.1 now follows by combining (4.20), (4.21), and (4.27).

5. Proof of Proposition 3.2

Define the functions

$$g_{jr}^{\pm}(\ell) = g_j^{\pm}(\ell/r) + \frac{\overline{ra_j}\ell}{q_j}, \quad h_{jr}^{\pm}(\ell) = \frac{\ell}{r}h_j^{\pm}(\ell/r),$$

$$H_{ijr}^{\pm}(\ell) = h_{ir}^{\pm}(\ell)h_{jr}^{\pm}(\ell), \quad \text{and} \quad W_{ijr}^{\pm}(\ell) = \omega_i\big(x_i^{\pm}(\ell/r)\big)\omega_j\big(x_j^{\pm}(\ell/r)\big),$$

where we recall that $g_j^{\pm}, h_j^{\pm}, x_j^{\pm}$, and ω_j are given by (3.14), (3.15), (3.13), and (3.8), respectively. Note that $|W_{ijr}^{\pm}(\ell)| \le 1$ and, since $x_j^{\pm}(\ell)$ is monotonic, $W_{ijr}^{\pm}(\ell)$ has bounded variation (over all of $\mathbb{R}$). Applying Cauchy's inequality in the ℓ variable in (3.18) and then expanding out the resulting square, we have

$$\begin{aligned}
&\Bigg|\sum_{\ell=L_1}^{L_2}\frac{r\lambda(\ell)}{\ell}\sum_{\substack{j\in J(\beta,r)\\(u_j,v_j)\in\mathcal{R}}}\nu(j)e\big(g_{jr}^{\pm}(\ell)\big)h_{jr}^{\pm}(\ell)\omega_j\big(x_j^{\pm}(\ell/r)\big)\Bigg|^2\\
&\quad\le r^2\sum_{\ell=L_1}^{L_2}\frac{|\lambda(\ell)|^2}{\ell^2}\sum_{\substack{i,j\in J(\beta,r)\\(u_i,v_i),(u_j,v_j)\in\mathcal{R}}}\nu(i)\overline{\nu(j)}\sum_{\ell=L_1}^{L_2}W_{ijr}^{\pm}(\ell)H_{ijr}^{\pm}(\ell)e\big(g_{ir}^{\pm}(\ell)-g_{jr}^{\pm}(\ell)\big)\\
&\quad\ll\frac{r^2}{L^2}\sum_{\ell=L_1}^{L_2}|\lambda(\ell)|^2\max_{j\in J(\beta,r)}|\nu(j)|^2\cdot\sum_{\substack{i,j\in J(\beta,r)\\(u_i,v_i),(u_j,v_j)\in\mathcal{R}}}\Bigg|\sum_{\ell=L_1}^{L_2}W_{ijr}^{\pm}(\ell)H_{ijr}^{\pm}(\ell)e\big(g_{ir}^{\pm}(\ell)-g_{jr}^{\pm}(\ell)\big)\Bigg|.
\end{aligned}$$

Let

$$Y_{ijr}^{\pm}(L_1,L_2) = \big|W_{ijr}^{\pm}(L_1)H_{ijr}^{\pm}(L_1)\big| + \int_{L_1}^{L_2}\left|\frac{\mathrm{d}}{\mathrm{d}x}W_{ijr}^{\pm}(x)H_{ijr}^{\pm}(x)\right|\mathrm{d}x.$$

We now apply [12, Lemma 5.1.1] to see that

$$\left|\sum_{\ell=L_1}^{L_2} W_{ijr}^{\pm}(\ell)H_{ijr}^{\pm}(\ell)e\big(g_{ir}^{\pm}(\ell)-g_{jr}^{\pm}(\ell)\big)\right| \;\le\; Y_{ijr}^{\pm}(L_1,L_2)\max_{L_1'\in[L_1,L_2]}\left|\sum_{\ell=L_1'}^{L_2} e\big(g_{ir}^{\pm}(\ell)-g_{jr}^{\pm}(\ell)\big)\right|.$$

We first estimate $Y_{ijr}^{\pm}(L_1,L_2)$. For any function W of bounded variation, we have

$$\int\left|\frac{\mathrm{d}}{\mathrm{d}x}(WH)\right|\mathrm{d}x \ll \max|H| + \int|H'(x)|\,\mathrm{d}x.$$

Therefore

$$Y_{ijr}^{\pm}(L_1,L_2) \ll \max_{\ell\in[L_1,L_2]} H_{ijr}^{\pm}(\ell) + \int_{L_1}^{L_2}\left|\frac{\mathrm{d}}{\mathrm{d}x}H_{ijr}^{\pm}(x)\right|\mathrm{d}x,$$

since $H_{ijr}^{\pm}(\ell)$ is positive. We now estimate the right-hand side of this inequality. Observe that

$$h_{jr}^{\pm}(\ell) = q_j\sqrt{\frac{\ell}{rq_j^2}x_j^{\pm}(\ell/r)}\cdot\left[\frac{t}{\pi\sqrt{\frac{\ell}{rq_j^2}x_j^{\pm}(\ell/r)}}\pm 1\right]^{-\frac{1}{2}}.$$

Let $y_j=\frac{d}{2ru_jq_j}$. Then

$$\sqrt{\frac{\ell}{rq_j^2}x_j^{\pm}(\ell/r)} = \sqrt{(\ell y_j)^2+\frac{t\ell y_j}{\pi}}\pm\ell y_j = \frac{\frac{ty_j}{\pi}}{\sqrt{y_j^2+\frac{ty_j}{\pi\ell}}\mp y_j}.$$

From the third formula, we see that this is an increasing function of ℓ for $\ell>0$ and thus $h_{jr}^{\pm}(\ell)$ is an increasing function of ℓ for $\ell>0$, as well. Therefore $Y_{ijr}^{\pm}(L_1,L_2)\ll H_{ijr}^{\pm}(L_2)$. Since $R=\sqrt{M/M_0}$ and $M\ll\sqrt{C}$, by (3.1) and (3.5) we have $M\ll tR$. This implies that

$$d^2L \ll \frac{rV^2M}{R^2} \asymp \frac{rUVM^2}{tR^2} \ll rtUV$$

so that $\frac{t}{\pi\ell y_j}\gg 1$ for all j. Therefore

$$\sqrt{\frac{\ell}{rq_j^2}x_j^{\pm}(\ell/r)} = \sqrt{(\ell y_j)^2+\frac{t\ell y_j}{\pi}}\pm\ell y_j \asymp \eta L$$

and thus

$$h_{jr}^{\pm}(\ell) \asymp \frac{V}{d}\sqrt{\frac{(\eta L)^3}{t}}.$$

This means that

$$Y_{ijr}^{\pm}(L_1,L_2) \ll \frac{V^2}{d^2t}(\eta L)^3 = \frac{\eta VL^2}{rU}.$$

We now estimate

$$\Sigma_{ijr}^{\pm}(L_1,L_2) := \max_{L_1\le L_1'\le L_2}\left|S_{ijr}^{\pm}(L_1',L_2)\right|$$

where

$$S_{ijr}^{\pm}(L_1',L_2) := \sum_{\ell=L_1'}^{L_2} e\left(g_i^{\pm}(\ell/r)-g_j^{\pm}(\ell/r)\right).$$

In order to estimate the exponential sum $S_{ijr}^{\pm}(L_1', L_2)$, we use van der Corput first and second derivative estimates in the form of [12, Lemmas 5.1.2 and 5.1.3]. In particular, we need to study the derivatives of the functions $g_{jr}^{\pm}(\ell)$.

5.1. **First derivative estimate.** Note that

$$\frac{\mathrm{d}}{\mathrm{d}\ell} g_{jr}^{\pm}(\ell) = \frac{1}{r}(g_j^{\pm})'(\ell/r) + \frac{\overline{ra_j}}{q_j}.$$

For the stationary point $x_j^{\pm}(\ell)$, we have

$$-\frac{t}{2\pi x_j^{\pm}(\ell)} + \frac{u_j}{v_j} \mp \frac{1}{q_j}\sqrt{\frac{\ell}{x_j^{\pm}(\ell)}} = 0,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}\ell} g_j^{\pm}(\ell) = \left\{-\frac{t}{2\pi x_j^{\pm}(\ell)} + \frac{u_j}{v_j} \mp \frac{1}{q_j}\sqrt{\frac{\ell}{x_j^{\pm}(\ell)}}\right\}\frac{\mathrm{d}x_j^{\pm}(\ell)}{\mathrm{d}\ell} \mp \frac{1}{q_j}\sqrt{\frac{x_j^{\pm}(\ell)}{\ell}} = \mp\frac{1}{q_j}\sqrt{\frac{x_j^{\pm}(\ell)}{\ell}}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\ell} g_{jr}^{\pm}(\ell) = \mp\frac{1}{rq_j}\sqrt{\frac{x_j^{\pm}(\ell/r)}{\ell/r}} + \frac{\overline{ra_j}}{q_j}. \tag{5.1}$$

5.2. **Second derivative estimate.** We have

$$\frac{\mathrm{d}^2}{\mathrm{d}\ell^2} g_{jr}^{\pm}(\ell) = \mp\frac{1}{rq_j}\frac{\mathrm{d}}{\mathrm{d}\ell}\sqrt{\frac{x_j^{\pm}(\ell/r)}{\ell/r}}.$$

Again writing $y_j = \frac{d}{2ru_jq_j}$, we have

$$\frac{1}{rq_j}\sqrt{\frac{x_j^{\pm}(\ell/r)}{\ell/r}} = \frac{d}{2ru_jq_j}\left(\sqrt{1+\frac{2tru_jq_j}{\pi d\ell}} \pm 1\right) = y_j\left(\sqrt{1+\frac{t}{\pi y_j \ell}} \pm 1\right),$$

and so

$$\frac{\mathrm{d}^2}{\mathrm{d}\ell^2} g_{jr}^{\pm}(\ell) = \mp\frac{1}{rq_j}\frac{\mathrm{d}}{\mathrm{d}\ell}\sqrt{\frac{x_j^{\pm}(\ell/r)}{\ell/r}} = \pm\frac{t}{2\pi\ell^2}\left(1+\frac{t}{\pi\ell y_j}\right)^{-\frac{1}{2}}.$$

This is clearly a monotonic function of y_j. Therefore $(g_{ir}^{\pm}-g_{jr}^{\pm})''(\ell)$ is either identically zero, or it is never zero, so $(g_{ir}^{\pm}-g_{jr}^{\pm})'(\ell)$ is monotone in ℓ. Hence

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\ell^2}\left[g_{ir}^{\pm}(\ell)-g_{jr}^{\pm}(\ell)\right] &= \pm\frac{t}{2\pi\ell^2}\left\{\frac{1}{\sqrt{1+\frac{t}{\pi\ell y_i}}}-\frac{1}{\sqrt{1+\frac{t}{\pi\ell y_j}}}\right\}\\ &= \pm\frac{t}{2\pi\ell^2}\left\{\frac{\frac{t}{\pi\ell y_j}-\frac{t}{\pi\ell y_i}}{\sqrt{1+\frac{t}{\pi\ell y_i}}\sqrt{1+\frac{t}{\pi\ell y_j}}\left(\sqrt{1+\frac{t}{\pi\ell y_i}}+\sqrt{1+\frac{t}{\pi\ell y_j}}\right)}\right\}\\ &= \pm\frac{rt^2}{\pi^2 d\ell^3}\left\{\frac{u_jq_j-u_iq_i}{\sqrt{1+\frac{t}{\pi\ell y_i}}\sqrt{1+\frac{t}{\pi\ell y_j}}\left(\sqrt{1+\frac{t}{\pi\ell y_i}}+\sqrt{1+\frac{t}{\pi\ell y_j}}\right)}\right\}.\end{aligned}$$

As shown above, we have $\frac{t}{\pi\ell y_j}\gg 1$, so that

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}\ell^2}\left[g_{ir}^{\pm}(\ell)-g_{jr}^{\pm}(\ell)\right]\right| \asymp \frac{rt^2}{dL^3}\frac{|u_iq_i-u_jq_j|}{(\frac{rtUV}{d^2L})^{\frac{3}{2}}} \asymp \frac{\eta}{L}\frac{|u_iv_i-u_jv_j|}{UV}.$$

5.3. **Applying van der Corput estimates.** Our strategy is to use both van der Corput first and second derivative estimates. If neither of these estimates is small, then this implies constraints on the sizes of $|u_iv_i-u_jv_j|$ and $\|\frac{\overline{ra_i}}{q_i}-\frac{\overline{ra_j}}{q_j}\|$. This leads to the counting problem given in Proposition 3.2. To this end, define

$$N(X)=\#\{(i,j)\in J(\beta,r)^2:(u_i,v_i),(u_j,v_j)\in\mathcal{R}\text{ and }\Sigma_{ijr}^{\pm}(L_1,L_2)\geq X\}.$$

Trivially $\Sigma_{ijr}^{\pm}(L_1,L_2)\leq L$, so we have

$$\begin{aligned}\sum_{\substack{(i,j)\in J(\beta,r)^2\\(u_i,v_i),(u_j,v_j)\in\mathcal{R}}}\Sigma_{ijr}^{\pm}(L_1,L_2) &= -\int_0^L X\,\mathrm{d}N(X)=\int_0^L N(X)\,\mathrm{d}X\\ &\leq X_1(\#\mathcal{R})^2+\int_{X_1}^L N(X)\,\mathrm{d}X,\end{aligned}\tag{5.2}$$

where we take $X_1:=A\sqrt{L}\max(\sqrt{\eta},1)$ for a sufficiently large constant A.

Suppose that $\Sigma_{ijr}^{\pm}(L_1,L_2)\geq X\geq X_1$. If $u_iv_i-u_jv_j\neq 0$ then we get a bound for $\Sigma_{ijr}^{\pm}$ by the second derivative test. In particular, by [12, Lemma 5.1.3], we have

$$X\leq\Sigma_{ijr}^{\pm}(L_1,L_2)\ll L\sqrt{\lambda_2}+\frac{1}{\sqrt{\lambda_2}}\ll\sqrt{\eta L}+\frac{1}{\sqrt{\lambda_2}}\leq\frac{X}{A}+\frac{1}{\sqrt{\lambda_2}},$$

where $\lambda_2=\frac{\eta}{L}\frac{|u_iv_i-u_jv_j|}{UV}$. If A is large enough, then this implies that $1/\sqrt{\lambda_2}\gg X$ so therefore $\lambda_2\ll X^{-2}$. Thus, we have

$$\left|u_iv_i-u_jv_j\right|\ll UV\frac{L}{\eta X^2},\tag{5.3}$$

and obviously this holds also when $u_iv_i-u_jv_j=0$.

Next we apply a first derivative estimate. Let $z_{ijr} = \frac{\overline{a_i r}}{q_i} - \frac{\overline{a_j r}}{q_j}$. Then, with y_j as above, from (5.1) we derive that

$$\left|\frac{\mathrm{d}}{\mathrm{d}\ell}\left[g_{ir}^{\pm}(\ell) - g_{jr}^{\pm}(\ell)\right] - z_{ijr}\right| = \left|\left(\sqrt{y_i^2 + \frac{ty_i}{\pi\ell}} \pm y_i\right) - \left(\sqrt{y_j^2 + \frac{ty_j}{\pi\ell}} \pm y_j\right)\right|$$

$$(5.4) \qquad = |y_i - y_j| \left|\frac{y_i + y_j + \frac{t}{\pi\ell}}{\sqrt{y_i^2 + \frac{ty_i}{\pi\ell}} + \sqrt{y_j^2 + \frac{ty_j}{\pi\ell}}} \pm 1\right|$$

$$\asymp |y_i - y_j| \sqrt{\frac{rUVT}{d^2 L}} \asymp \eta \frac{|u_i v_i - u_j v_j|}{UV}.$$

Therefore, using the upper bound for $|u_i v_i - u_j v_j|$ in (5.3), we have

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\left[g_{ir}^{\pm}(\ell) - g_{jr}^{\pm}(\ell)\right] - z_{ijr} \ll \frac{L}{X^2} \leq \frac{L}{X_1^2} \ll 1.$$

By (possibly) increasing the size of the constant A in the definition of X_1, we see that there exist numbers μ and ν such that $\nu - \mu < \frac{1}{2}$ and $\mu \leq \frac{\mathrm{d}}{\mathrm{d}\ell}\left[g_{ir}^{\pm}(\ell) - g_{jr}^{\pm}(\ell)\right] \leq \nu$. Now we use the truncated Poisson summation formula [12, Lemma 5.4.3], which states that

$$\sum_{\ell = L_1'}^{L_2} e\big(g_{ir}^{\pm}(\ell) - g_{jr}^{\pm}(\ell)\big) = \sum_{\mu - \frac{1}{4} \leq n \leq \nu + \frac{1}{4}} \int_{L_1'}^{L_2} e\big(g_{ir}^{\pm}(x) - g_{jr}^{\pm}(x) - nx\big)\,\mathrm{d}x + O(1).$$

Note that the sum on the right-hand side contains at most one term. Hence

$$\Sigma_{ijr}^{\pm}(L_1, L_2) \leq O(1) + \max_{L_1' \in [L_1, L_2]} \sum_{\mu - \frac{1}{4} \leq n \leq \nu + \frac{1}{4}} \left|\int_{L_1'}^{L_2} e\big(g_{ir}^{\pm}(x) - g_{jr}^{\pm}(x) - nx\big)\,\mathrm{d}x\right|$$

$$= O(1) + \sum_{\mu - \frac{1}{4} \leq n \leq \nu + \frac{1}{4}} \max_{L_1' \in [L_1, L_2]} \left|\int_{L_1'}^{L_2} e\big(g_{ir}^{\pm}(x) - g_{jr}^{\pm}(x) - nx\big)\,\mathrm{d}x\right|.$$

Since z_{ijr} is only defined modulo 1, we are free to shift $g_{ir}^{\pm}(\ell) - g_{jr}^{\pm}(\ell)$ by any integer multiple of ℓ. Thus, if there is an integer $n \in [\mu - \frac{1}{4}, \nu + \frac{1}{4}]$, we may assume that $n = 0$. Therefore,

$$\Sigma_{ijr}^{\pm}(L_1, L_2) \leq \max_{L_1' \in [L_1, L_2]} \left|\int_{L_1'}^{L_2} e\big(g_{ir}^{\pm}(x) - g_{jr}^{\pm}(x)\big)\,\mathrm{d}x\right| + O(1).$$

Define

$$\lambda_1 = \min_{\ell \in [L_1, L_2]} \left|\frac{\mathrm{d}}{\mathrm{d}\ell}\left[g_{ir}^{\pm}(\ell) - g_{jr}^{\pm}(\ell)\right]\right|.$$

Then by [12, Lemma 5.1.2] we have

$$\max_{L_1' \in [L_1, L_2]} \left|\int_{L_1'}^{L_2} e\big(g_{ir}^{\pm}(x) - g_{jr}^{\pm}(x)\big)\,\mathrm{d}x\right| \ll \frac{1}{\lambda_1}.$$

This implies that $X \le \Sigma_{ijr}^{\pm}(L_1, L_2) \ll \frac{1}{\lambda_1} + 1$, so that $\lambda_1 \ll \frac{1}{X}$ if the constant A is sufficiently large. Thus, by (5.4), we find that

$$\|z_{ijr}\| \le |z_{ijr}| = \lambda_1 + O\left(\eta \frac{|u_i v_i - u_j v_j|}{UV}\right) \ll \frac{1}{X} + \frac{L}{X^2} \ll \frac{L}{X^2}, \tag{5.5}$$

since $X \le L$.

In summary, we have found that $\Sigma_{ijr}^{\pm}(L_1, L_2) \ge X \ge X_1$ implies the inequalities (5.3) and (5.5). In other words, $N(X) \le B(\Delta_1(X), \Delta_2(X))$ for certain functions $\Delta_1(X)$ and $\Delta_2(X)$ satisfying the conditions in (3.19). From (5.2), we derive that

$$\sum_{\substack{(i,j)\in J(\beta,r)^2 \\ (u_i,v_i),(u_j,v_j)\in\mathcal{R}}} \Sigma_{ijr}^{\pm}(L_1, L_2) \le X_1(\#\mathcal{R})^2 + \int_{X_1}^{L} B(\Delta_1(X)\Delta_2(X))\,\mathrm{d}X.$$

By extending the definitions of $\Delta_1(X)$ and $\Delta_2(X)$ to be zero for $X < X_1$, we note that the right-hand side of this expression is

$$\ll X_0(\#\mathcal{R})^2 + \int_{X_0}^{L} B(\Delta_1(X), \Delta_2(X))\,\mathrm{d}X$$

for X_0 defined in Proposition 3.2 so long as $A \ge 1$. The proposition now follows.

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